Validation for scientific computations
Stochastic arithmetic

Cours de recherche master informatique

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References for today's lecture

- J.-M. Chesneaux : Estimation statistique des erreurs d'arrondi, in Outils d'analyse numérique pour l'automatique, A. Barraud et al., Hermes, 2002

- D. Stott Parker: Monte Carlo arithmetic, several papers available from http://www.cs.ucla.edu/~stott/mca/

- W. Kahan: The improbability of probabilistic error analyses for numerical computations, 1998, http://http.cs.berkeley.edu/~wkahan/improber.ps

Agenda

- Introduction to stochastic arithmetic (CESTAC method, CADNA implementation)
- very brief introduction to Monte Carlo arithmetic
- comments by W. Kahan

Monte Carlo arithmetic

Exact value: a real number that can be exactly represented in a given floating-point format.

Inexact value: either a real number that cannot be exactly represented in a given floating-point format and thus must be rounded,
or a real value that is not completely known.

Monte Carlo arithmetic

Essence: model any inexact value with a random variable.

Randomization of x to s digits:

$$\tilde{x} = x + 2^{e+1-s}\zeta$$

where e is the exponent of x in base 2,
s is a real value (typically a positive integer)
ζ is a random variable (typically uniform over $[-\frac{1}{2}, \frac{1}{2}]$).

Monte Carlo arithmetic

$$\text{randomize}(x) = \begin{cases} x & \text{if } x \text{ is exact within } t \text{ digits} \\ x + 2^{e+1-t}\zeta & \text{otherwise.} \end{cases}$$

If x is not exact within t digits, this superimposes a random perturbation so that the resulting significance is bounded by t digits.

Monte Carlo arithmetic

random rounding:

random_round (x) = round (randomize (x)).

random unrounding:

$x \odot y$ = round (randomize $(x) \bullet$ randomize (y)).

Monte Carlo arithmetic

Monte Carlo arithmetic operations:

$x \odot y =$ round (randomize(randomize $(x)\bullet$ randomize (y))).

Implementation???

W. Kahan's analysis
Definition of error analysis

Definition of error analysis:
it is a process and a product; it is an estimate of the error in a computation, and a proof of the estimate's validity.

W. Kahan's analysis
Error analyses

Error analyses start from models of errors, like β and μ in

$$w = ((x \cdot y) \cdot (1 + \beta) + z) \cdot (1 + \mu),$$

taking more or less their properties into account, to infer estimates of their subsequent effects. Some analyses obscure their domains of validity by ignoring nonlinear terms like β^2, $\beta \cdot \mu$ and μ^2. Anyway, inferences entail tedious manipulations of numerous inequalities, only partly mechanizable.

Probabilistic error analyses estimate errors' means and standard deviations instead of upper bounds for errors.

W. Kahan's analysis

Two statistical strategies: Theoretical and Experimental

Theoretical: Probabilistic Error-Analyses

. . . are based upon attempts to approximate each rounding error by a random variate of tiny amplitude, and then estimates how lots of them will propagate and accumulate in the final computed results.

W. Kahan's analysis

Two statistical strategies: Theoretical and Experimental

Experimental: Randomized Error-Sampling

. . . attempts to assay the impact of roundoff upon any computation by treating that computation as one sample drawn from a population of similar randomized computations differing only

- either in the **data**, which are randomly perturbed slightly from the given data (F. Chatelin and V. Frayssé)
- or in **arithmetic operations**, which are randomly perturbed slightly (J. Vignes et al.).

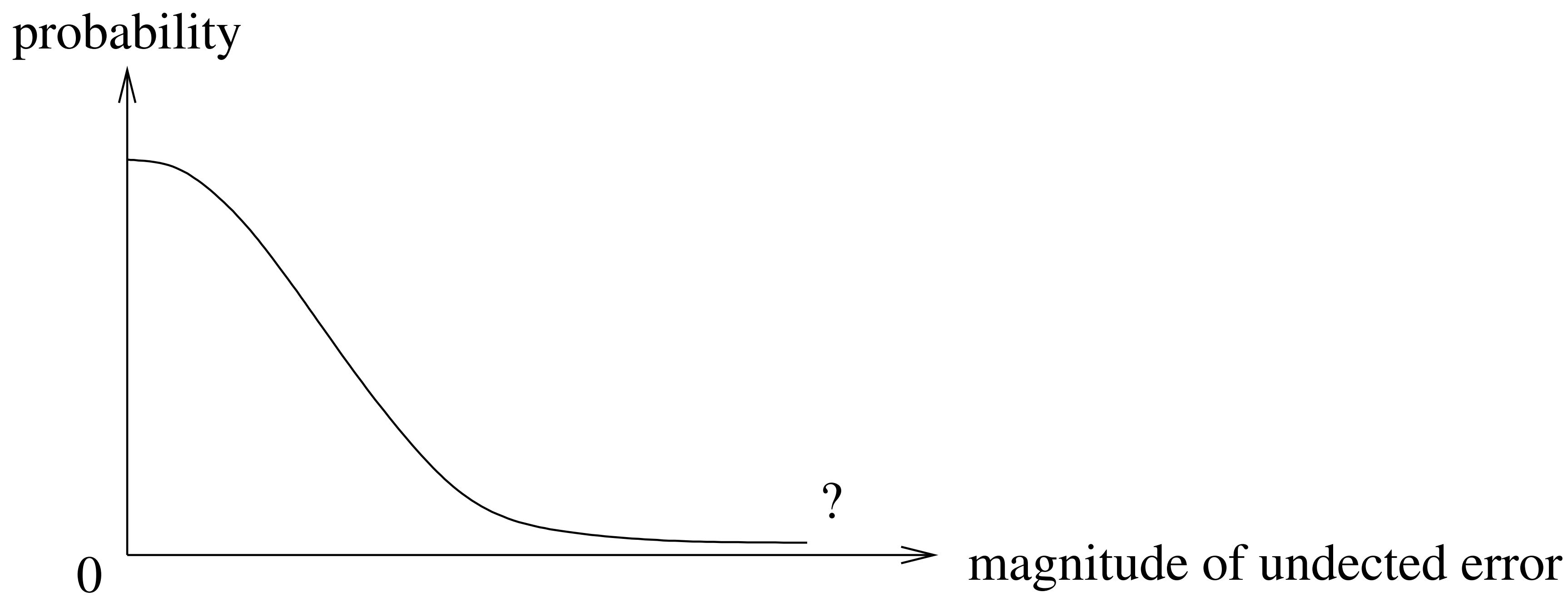
W. Kahan's analysis
why are they unsatisfactory?

They tend to provide unsatisfactory answers for two crucial questions:

1. **insurance premiums:** how much should a prudent corporation put into reserve to cover the expected cost of extraordinarily big numerical errors detexted too late?

2. **unreliability:** how likely is an extraordinarily big numerical error, if one occurs, to be detected too late despite probabilistic analysis and/or randomized error sampling?

W. Kahan's analysis problem



Three reasons, all of which call into question the application of the **Central Limit Theorem**.

W. Kahan's analysis

1st problem with the application of the Central Limit Theorem

The Central Limit Theorem is generally cited to justify approximating probability via a normal or ξ^2 distribution. But such approximations converge very slowly along the tails of the distribution. Therefore, where probability is tiny, the approximation can be extremely tiny and yet wrong by orders of magnitude.

W. Kahan's analysis

2nd problem with the application of the Central Limit Theorem

To justify invoking this theorem, rounding errors are presumed to be

- random
- weakly correlated
- distributed continuously over a tiny interval.

Actually, they are

- not random
- often correlated (perhaps intentionally)
- often behave more like discrete than continuous variables.

W. Kahan's analysis

2nd problem with the application of the Central Limit Theorem

Illustration: cf. the example with the rational fraction from his talk "Improber".

Illustration: cf. lecture 2: Sterbenz lemma (no rounding error, please do not introduce one), or $\frac{x}{\sqrt{x^2+y^2}}$

W. Kahan's analysis

3rd problem with the application of the Central Limit Theorem

Only a few (as few as two or three) rounding errors are the dominant contributors to the final error, especially when it is extraordinarily big because some nearby singularity amplified them.

Examples:
cf. $3 * tan(atan(10000000.0))/10000000.0)$ and the "singularity" of atan
cf. the rational fraction: the first subtractions performed in the numerator and denominator of rp(x) contribute two rounding errors that dominate all the rest.

W. Kahan's analysis
example of failures

Solve

$$\begin{pmatrix} 4194304 & 4194303 & & & & \\ 4194303 & 4194302 & & & & \\ & & 4194304 & 4194303 & & \\ & & 4194303 & 4194302 & & \\ & & & & 4194304 & 4194303 \\ & & & & 4194303 & 4194302 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \\ 0 \\ 3 \end{pmatrix}$$

Solutions to each block system:

$(1.3E+7 \quad -1.3E+7)^t$ for the first block

$(1.2E+7 \quad -1.2E+7)^t$ for the second block

$(1.2509611E+7 \quad -1.2509613E+7)^t$ for the third block.

Further readings

- F. Chatelin and V. Frayssé: Lecture on Finite Precision Arithmetic, SIAM, 1996

- M. Daumas and D. Lester: Formal Methods for Rare Failure Events due to the Accumulation of Errors, Oct. 2006, http://arXiv.org/abs/cs/0610110.