Problem 1

Let $\widetilde{\Sigma} = (\{a,b\},\{b,c\},\{d\})$ be a distributed alphabet. Consider the following subsets of Σ^*:

$$L_1 = \{w \in \Sigma^* \mid |w|_a = |w|_c\}$$

$$L_2 = \{w \in \Sigma^* \mid |w|_a + |w|_d \text{ is even}\}$$

$$L_3 = \{uadcv \mid u, v \in \Sigma^*\}$$

For $i = 1, 2, 3$, determine if

(a) $L_i = [L_i]_{\sim_{\widetilde{\Sigma}}}$ (recall that $[L_i]_{\sim_{\widetilde{\Sigma}}} := \{u \in \Sigma^* \mid \text{there is } v \in L_i \text{ such that } u \sim_{\widetilde{\Sigma}} v\}$).

(b) $[L_i]_{\sim_{\widetilde{\Sigma}}}$ is regular.

(c) there is an asynchronous automaton $\mathcal{A}$ over $\widetilde{\Sigma}$ such that $L(\mathcal{A}) = [L_i]_{\sim_{\widetilde{\Sigma}}}$.

(d) there is a product automaton $\mathcal{A}$ over $\widetilde{\Sigma}$ such that $L(\mathcal{A}) = [L_i]_{\sim_{\widetilde{\Sigma}}}$.

(e) there is a locally accepting product automaton $\mathcal{A}$ over $\widetilde{\Sigma}$ such that $L(\mathcal{A}) = [L_i]_{\sim_{\widetilde{\Sigma}}}$.

Justify your answers!

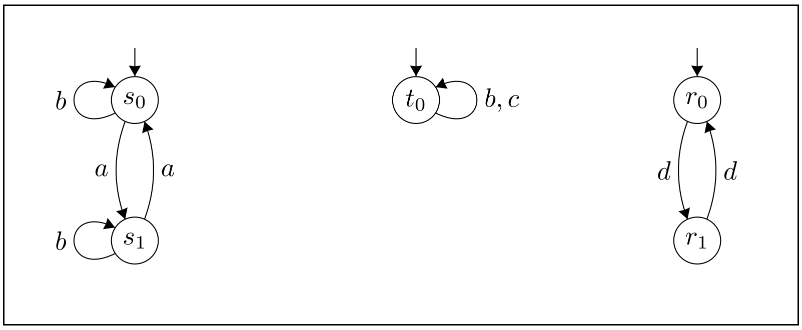
(Recall that, for $w \in \Sigma^*$ and $\sigma \in \Sigma$, $|w|_\sigma$ denotes the number of occurrences of σ in w. For example, $|abaac|_a = 3$ and $|abc|_d = 0$.)

Solution

	(a)	(b)	(c)	(d)	(e)
1	✓	×	×	×	×
2	✓	✓	✓	✓	×
3	×	✓	✓	×	×

Note that statements (b) and (c) are equivalent!

A product automaton for L_2 is given as follows:



with set of global final states $\{(s_0, t_0, r_0), (s_1, t_0, r_1)\}$

Note that any *locally accepting* product automaton over $\widetilde{\Sigma}$ that accepts both ad and $aadd$ also accepts $add \notin L_2$.

Problem 2

Let $\widetilde{\Sigma} = (\{a,b\},\{b,c\},\{d\})$ be a distributed alphabet. For any of the following rational expressions $\alpha_i \in \mathrm{REX}_{\mathbb{TR}(\widetilde{\Sigma})}$ $(i = 1,2,3)$, examine if α_i is star-connected. If not, is there a star-connected rational expression $\beta \in \mathrm{REX}_{\mathbb{TR}(\widetilde{\Sigma})}$ such that $\mathcal{L}(\beta) = \mathcal{L}(\alpha_i)$?

$\alpha_1 = \emptyset \cdot (a \cdot c)^*$

$\alpha_2 = (a + c^*)^* \cdot d$

$\alpha_3 = (a \cdot (c \cdot b^*)^*)^*$

<u>*Solution*</u>

α_1 is *not* star-connected. But $\emptyset$, which is star-connected, is equivalent to α_1.

α_2 is star-connected.

α_3 is *not* star-connected, because $\mathcal{L}(a \cdot (c \cdot b^*)^*)$ contains traces that are not connected. However, the star-connected rational expression $\emptyset^* + a \cdot (a + c \cdot b^*)^*$ is equivalent to α_3.

Problem 3

Let $Proc = \{1,2,3,4\}$ and let Msg be a singleton set.

(a) Determine a formula $\varphi(x,y) \in \mathrm{MSO}(Act, \{\le, <_{\mathrm{mes}}\})$ with free variables x and y such that, for any $\mathcal{M} = (E, \le, \lambda) \in \mathbb{MSC}$ and any interpretation function $\mathcal{I}$,

$$\mathcal{M} \models_{\mathcal{I}} \varphi(x,y) \;\text{ iff }\; (\mathcal{I}(x), \mathcal{I}(y)) \in (\le \cup \le^{-1})^*.$$

(b) Determine a sentence $\varphi \in \mathrm{MSO}(Act, \{\le, <_{\mathrm{mes}}\})$ such that

$$\mathcal{L}_{\mathbb{MSC}}(\varphi) = \{\mathcal{M} \in \mathbb{MSC} \mid \mathcal{M} \text{ is connected}\}.$$

Is there a rational expression $\alpha \in \mathrm{REX}_{\mathbb{MSC}}$ such that $\mathcal{L}(\alpha) = \mathcal{L}_{\mathbb{MSC}}(\varphi)$?

(c) Determine a star-connected rational expression $\alpha \in \mathrm{REX}_{\mathbb{MSC}}$ such that

$$\mathcal{L}(\alpha) = \{(E, \le, \lambda) \in \mathbb{MSC} \mid \text{for any } e \in E,\; \lambda(e) \in \{1!2, 2?1, 2!3, 3?2, 3!4, 4?3\}\}.$$

Is there an equivalent rational expression that is not star-connected?

(d) Determine a star-connected rational expression $\alpha \in \mathrm{REX}_{\mathbb{MSC}}$ such that

$$\mathcal{L}(\alpha) = \{(E, \le, \lambda) \in \mathbb{MSC} \mid \text{for any } e \in E,\; \lambda(e) \in \{1!2, 2?1, 2!3, 3?2\}\}.$$

Is there an equivalent rational expression that is not star-connected?

Solution

(a) The formula $\psi(x,y) = x \le y \;\vee\; y \le x$ expresses that x and y are ordered by $\le$. Now set $\varphi(x,y)$ to be

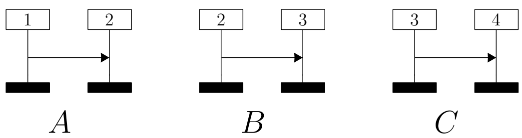
$$\exists z_1 \exists z_2 \exists z_3 \exists z_4 \Big(\psi(x, z_1) \wedge \psi(z_1, z_2) \wedge \psi(z_2, z_3) \wedge \psi(z_3, z_4) \wedge \psi(z_4, y) \Big)$$

(b) Let *connected*(x,y) be the formula from part (a). For

$$\varphi = \forall x \forall y \; \mathit{connected}(x,y)$$

we have $\mathcal{L}_{\mathbb{MSC}}(\varphi) = \{\mathcal{M} \in \mathbb{MSC} \mid \mathcal{M} \text{ is connected}\}$.

In the following, let



(c) $\alpha = (A + B + C)^*$
$\beta = (A + B + C + A \cdot C)^*$ is not star-connected but equivalent to α.

(d) $\alpha = (A + B)^*$
$\beta = (A + B)^* + \emptyset \cdot (A \cdot C)^*$ is not star-connected but equivalent to α.

Problem 4

Argue that the following problem is undecidable for *Proc* $= \{1, 2, 3\}$ and a singleton set *Msg*:

> Input: MPA $\mathcal{A}$ over *Proc* and *Msg*.
>
> Question: Is $\mathcal{A}$ $\exists B$-bounded for some $B \geq 1$?

Solution

The proof is by reduction from the halting problem for a Turing machine *TM*, say with final state q_f. We build an MPA $\mathcal{A}$ over $\{1, 2, 3\}$ and *Msg* such that $\mathcal{A}$ is not $\exists B$-bounded for any $B \geq 1$ iff *TM* can reach q_f: Process 1 will send arbitrarily many messages to process 2, followed by a message to process 3. Process 3 receives this message and sends a message two process 2. Process 2 must receive the message from process 3 before receiving all the messages that have been initially sent by process 1. So far, this describes a behavior that is not $\exists B$-bounded, no matter which $B \geq 1$ we choose. Once process 2 has received all its messages, processes 1 and 2 start simulating *TM* so that $\mathcal{A}$ is $\exists B$-bounded for some $B \geq 1$ iff $\mathcal{L}(\mathcal{A}) = \emptyset$ iff *TM* can *not* reach q_f. The idea is illustrated as follows: