Modeling and verifying reactive systems

Temporal logics

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Outline of the course

1. Branching-time temporal logics
   - Complexity
   - Alternating-time Temporal Logic

2. Timed temporal logics
   - Timed models
   - Timed logics
   - Undecidability

CTL* model-checking

Theorem

CTL model-checking is* PSPACE*-complete.*

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Proof.

- hardness in PSPACE: CTL* subsumes LTL.
- membership in PSPACE: labeling algorithm involving LTL model-checking algorithm.

□

ECTL^{+} model-checking

Theorem

ECTL^{+} model-checking is Δ_2^{P}-complete.

ECTL$^+$ model-checking

Theorem

ECTL$^+$ model-checking is Δ_2^P-complete.

Proof.

- hardness in NP: easy encoding of SAT as a CTL$^+$ model-checking problem.
  Hardness in Δ_2^P is an intricate extension of that encoding.
- membership in Δ_2^P: using an oracle for deciding LTL_1-subformulas;

□

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Multi-agent systems

Problem

The CTL formula

$$\mathbf{A}\,\mathbf{G}(\mathbf{E}\,\mathbf{F}\,\textit{cabin.ground floor})$$

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it is always possible to reach the ground floor.

We rather mean that there is a strategy that makes the cabin eventually reach the ground floor. Moreover, we'd prefer that this strategy only involves the button at the third floor (say) and the buttons in the cabin.

Multi-agent systems

Definition

A *CGS* $\mathcal{C}$ is a 6-tuple $(Q, \mathrm{AP}, \ell, \mathbb{A}, \mathrm{Mv}, \mathrm{Edg})$ s.t:

- Q: a finite set of *locations*;
- AP: a finite set of *atomic propositions*;
- $\ell \colon Q \to 2^{\mathrm{AP}}$: a labeling function;
- $\mathbb{A} = \{A_1, ..., A_k\}$: a set of *agents* (or *players*);
- $\mathrm{Mv} \colon Q \times \mathbb{A} \to \mathcal{P}(\mathbb{Z}^+)$ the choice function.
  $\mathrm{Mv}(\ell, A_i)$ = set of possible moves for player A_i from ℓ.
- $\mathrm{Edg} \colon Q \times \mathbb{Z}^{+k} \to Q$: the transition table.

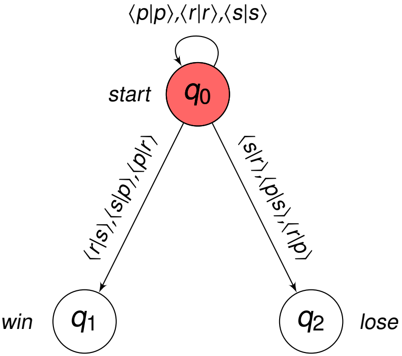
Semantics of CGSs

- From a location ℓ, each agent A_i chooses some m_{A_i} with

$$m_{A_i} \in \mathrm{Mv}(\ell, A_i).$$

- $\mathrm{Edg}(\ell, m_{A_1}, \cdots, m_{A_k})$ gives the new location.

Example

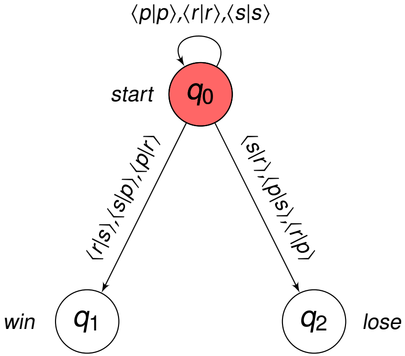


Player 2

Player 1

q_0	p	r	s
p	q_0	q_1	q_2
r	q_2	q_0	q_1
s	q_1	q_2	q_0

Example



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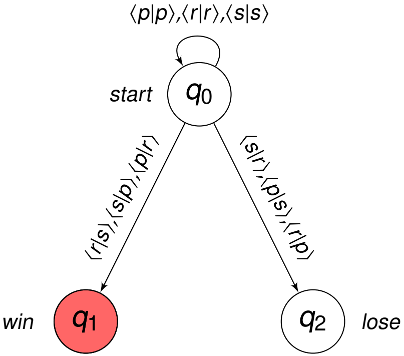

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Notations:

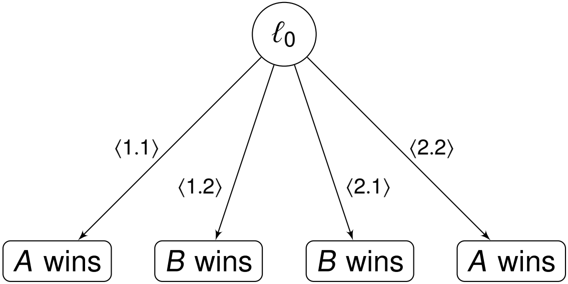
- $\mathrm{Next}(\ell) = \left\{\mathrm{Edg}(\ell, \cdots m_{A_i} \cdots) \mid \forall m_{A_i} \cdot 1 \leq i \leq k\right\}$
- $\mathrm{Next}(\ell, A_j, m) = \left\{\mathrm{Edg}(\ell, \cdots, m_{A_{j-1}}, m, m_{A_{j+1}}, \cdots)\right\}$

Strategies and outcomes

Definition

- A computation is an infinite sequence $\rho = \ell_0 \ell_1 \cdots$ such that $\forall i, \ell_{i+1} \in \text{Next}(\ell_i)$.
- A strategy for agent A_i is a function f_{A_i} s.t.
  $$f_{A_i}(\ell_0, \cdots, \ell_m) \in \text{Mv}(\ell_m, A_i).$$
- The outcomes $\text{Out}(\ell, f_{A_i})$ are the set of computations from ℓ that agree with the strategy f_{A_i} of A_i.
- Those notions extend to coallitions of agents: given $A \subseteq \mathbb{A}$, we write
  - $F_A = \{f_{A_i} | A_i \in A\}$
  - $\text{Out}(\ell, F_A)$

Another example



- player A has no strategy to win.
- player B has no strategy to win.

Synchronous games are not determined.

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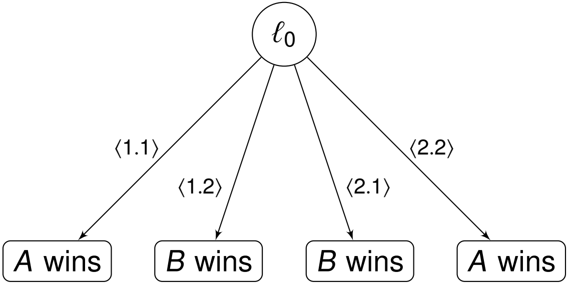

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Synchronous games are not determined.

Theorem (Martin, 1975)

Turn-based games (with reasonnable winning conditions) are determined.

Syntax of ATL

Definition

The syntax of ATL is defined by the following grammar:

$$\begin{aligned} \text{ATL} \ni \varphi_s, \psi_s &::= p \mid \neg \varphi_s \mid \varphi_s \vee \psi_s \mid \langle\!\langle A \rangle\!\rangle \varphi_p \mid [\![A]\!] \varphi_p \\ \varphi_p &::= \mathbf{X}\, \varphi_s \mid \varphi_s \,\mathbf{U}\, \psi_s. \end{aligned}$$

where p ranges over the set AP and A over the subsets of $\mathbb{A}$.

ATL subsumes CTL, since we have:

$$\mathbf{E}\varphi_p \equiv \langle\!\langle \mathbb{A} \rangle\!\rangle \varphi_p,$$

$$\mathbf{A}\varphi_p \equiv \langle\!\langle \varnothing \rangle\!\rangle \varphi_p.$$

Semantics of ATL

Definition

- Semantics

$$\ell \models \langle\!\langle A \rangle\!\rangle \varphi_p \quad \text{iff} \quad \exists F_A \in \text{Strat}(A).\ \forall \rho \in \text{Out}(\ell, F_A).\ \rho \models \varphi_p$$

$$\rho \models \varphi_s \;\mathbf{U}\; \psi_s \quad \text{iff} \quad \exists i. \rho[i] \models \psi_s \text{ and } \forall 0 \leq j < i. \rho[j] \models \varphi_s$$

$$\rho \models \mathbf{X}\, \varphi_s \quad \text{iff} \quad \rho[1] \models \varphi_s$$

- We have $\langle\!\langle A \rangle\!\rangle \varphi \Rightarrow \neg \langle\!\langle \mathbb{A} \setminus A \rangle\!\rangle \neg \varphi$,
  but

$$\neg \langle\!\langle A \rangle\!\rangle \varphi \not\Rightarrow \langle\!\langle \mathbb{A} \setminus A \rangle\!\rangle \neg\varphi.$$

- The semantics of $[\![A]\!]\varphi$ is that of $\neg \langle\!\langle A \rangle\!\rangle \neg\varphi$

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- we cannot restrict to modalities $\langle\langle A\rangle\rangle\mathbf{X}$, $\langle\langle A\rangle\rangle\mathbf{G}$ and $\langle\langle A\rangle\rangle\mathbf{U}$: modality $[[A]]\,\mathbf{U}$ cannot be expressed from those three modalities;

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- this algorithm runs in time $O(|\varphi| \cdot |{\rightarrow}|)$.

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Temporal logics = qualitative requirements

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Any request is granted in at most 1 minute.

An alarm rings if the doors are open for more than 30 seconds.

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An alarm rings if the doors are open for more than 30 seconds.

Requires explicit timing constraints in the model.

Adding “time” in Kripke structures

- basic idea: counting the number of transitions:

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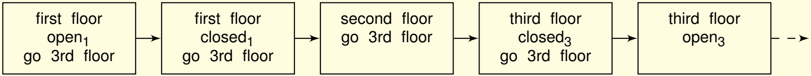
Examples



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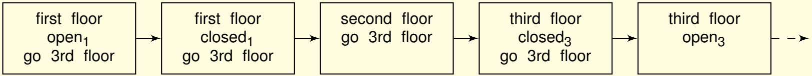

$$\mathbf{G}(\text{go 3rd floor} \Rightarrow \mathbf{F}_{\leq 4}\, \text{cabin.open}_3)$$

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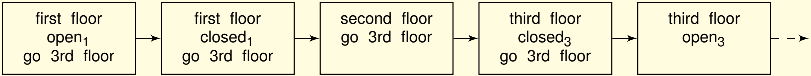

$$\mathbf{A}\,\mathbf{G}(\mathbf{E}\,\mathbf{F}_{\leq 10}\ \text{cabin.open}_1)$$

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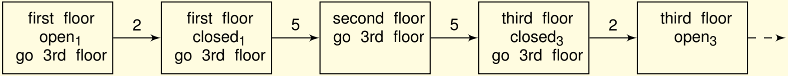

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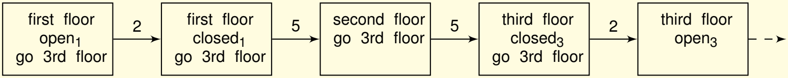

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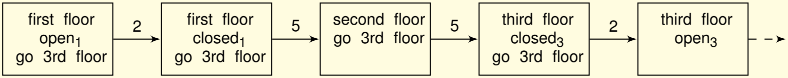

$$\mathbf{A}\,\mathbf{G}(\mathbf{E}\,\mathbf{F}_{\leq 25}\,\text{open}_1)$$

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$$\mathbf{A\,G}(\mathbf{E\,F}_{\leq 25}\,\text{open}_1) \equiv \mathbf{A\,G}(\mathbf{E\,X}(\text{open}_1 \vee \mathbf{E\,X}(\text{open}_1 \vee \mathbf{E\,X}(\text{open}_1 \vee \mathbf{E\,X}(\text{open}_1 \ldots)))))$$

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Theorem

Model-checking TCTL on timed Kripke structures is PSPACE*-complete.*
Model-checking TLTL on timed Kripke structures is EXPSPACE*-complete.*

Timed automata

Definition

A *timed automaton* is a tuple $\mathcal{A} = \langle Q, Q_0, C, \rightarrow, \Sigma, \ell \rangle$ s.t.:

- Q is the set of locations, of which Q_0 are initial;
- C is a (finite) set of *clock variables*;
- $\rightarrow$ is the set of transitions
- Σ is the alphabet;
- ℓ labels either the states or the transitions.

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Clocks are used on transitions: a transition is labeled with a *guard*, i.e., a list of constraints $x \sim n$ where $x \in \mathcal{C}$, $n \in \mathbb{Z}^+$ and $\sim \in \{<, \leq, =, \geq, >\}$.

Timed automata

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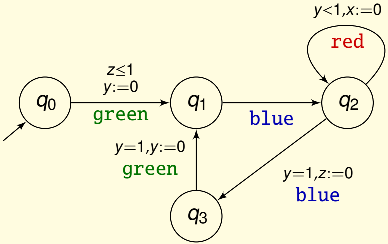

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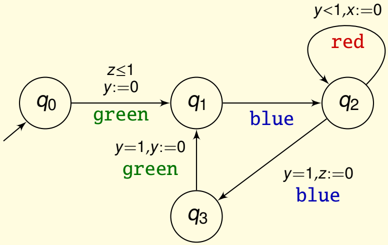

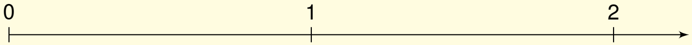

$x=$

$y=$

$z=$

Timed automata

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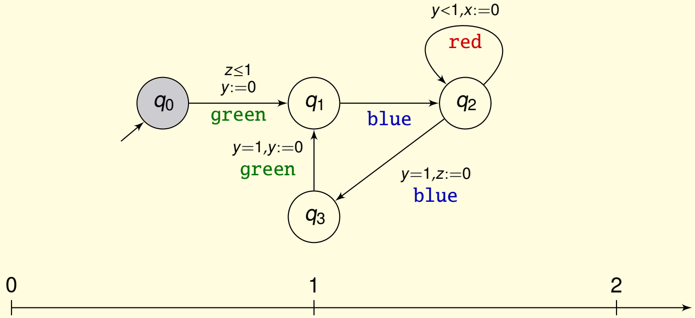

$x=$ 0

$y=$ 0

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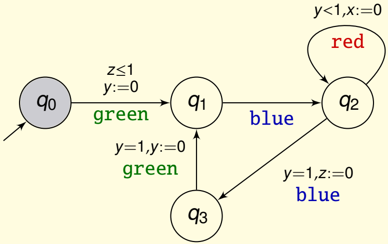
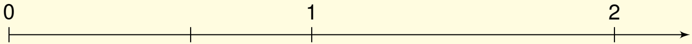
Timed automata

Example





$x=$	0	0.6
$y=$	0	0.6
$z=$	0	0.6

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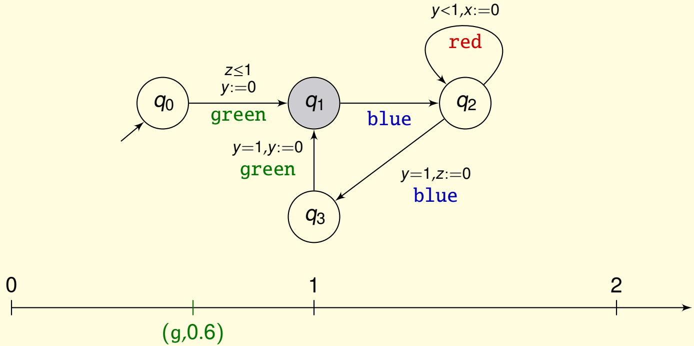

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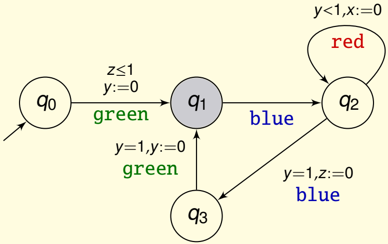

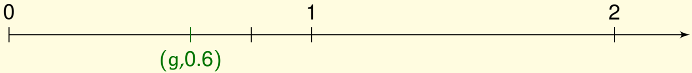

$x=$	0	0.6	0.8
$y=$	0	0	0.2
$z=$	0	0.6	0.8

Timed automata

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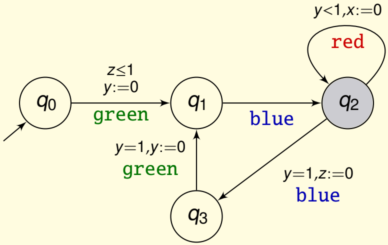

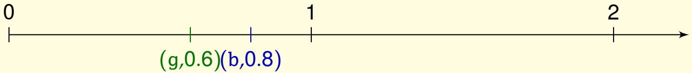

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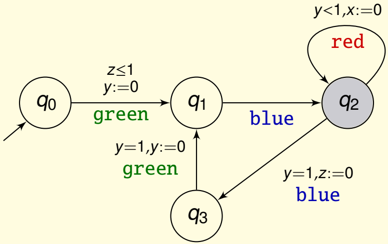

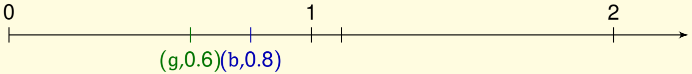

$x=$	0	0.6	0.8	1.1
$y=$	0	0	0.2	0.5
$z=$	0	0.6	0.8	1.1

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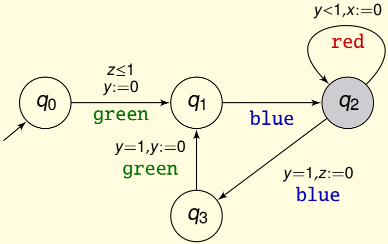

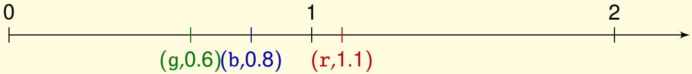

$x=$	0	0.6	0.8	0
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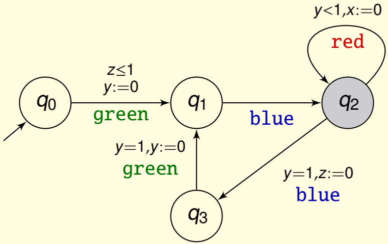

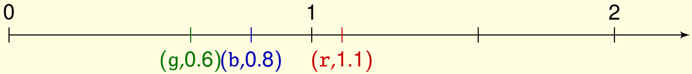

$x=$	0	0.6	0.8	0	0.5
$y=$	0	0	0.2	0.5	1
$z=$	0	0.6	0.8	1.1	1.6

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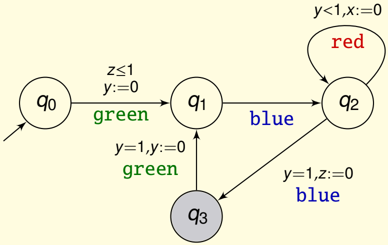

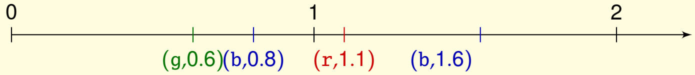

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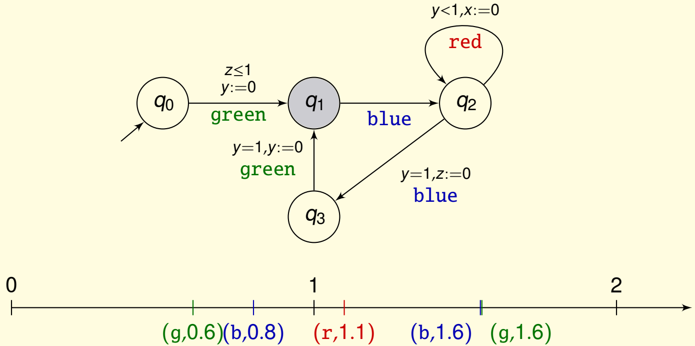

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$$w = (\texttt{green}, 0.6)\,(\texttt{blue}, 0.8)\,(\texttt{red}, 1.1)\,(\texttt{blue}, 1.6)\,(\texttt{green}, 1.6)\ldots$$

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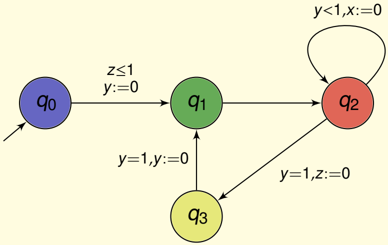
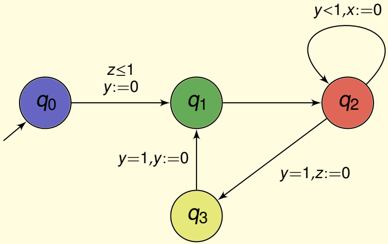
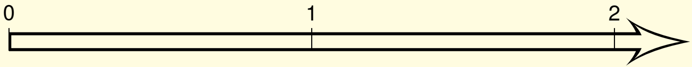


Timed automata

Example





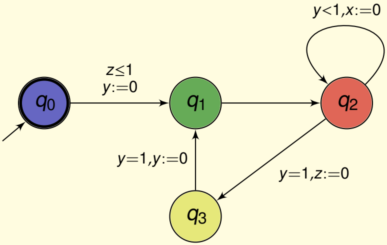
$x=$

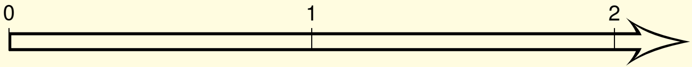
$y=$

$z=$

Timed automata

Example





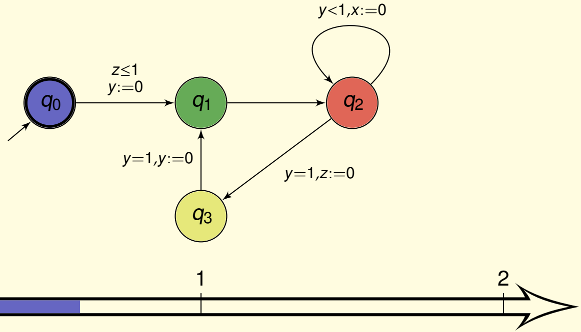
$x =$ 0

$y =$ 0

$z =$ 0

Timed automata

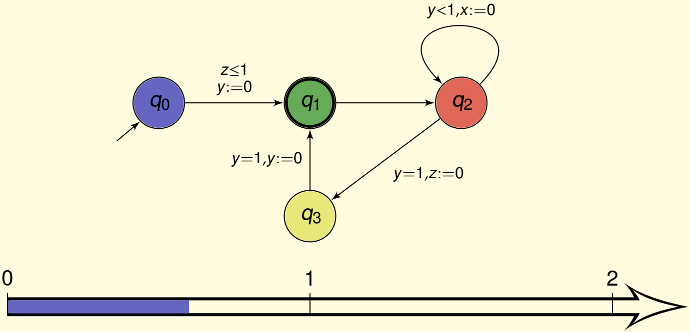
Example



$x=$	0	0.6
$y=$	0	0.6
$z=$	0	0.6

Timed automata

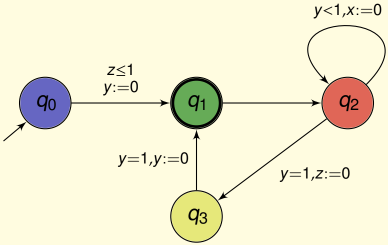
Example

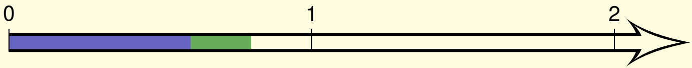


$x=$	0	0.6
$y=$	0	0
$z=$	0	0.6

Timed automata

Example

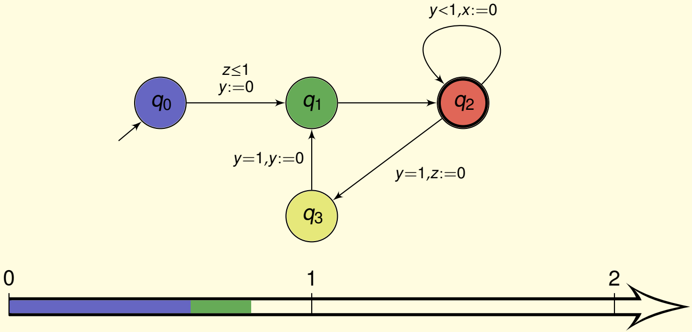




$x=$	0	0.6	0.8
$y=$	0	0	0.2
$z=$	0	0.6	0.8

Timed automata

Example



$x=$	0	0.6	0.8
$y=$	0	0	0.2
$z=$	0	0.6	0.8

Timed automata

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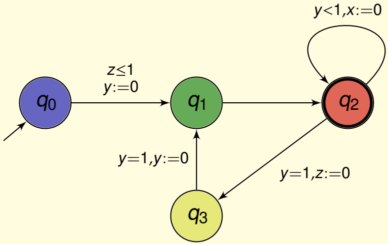
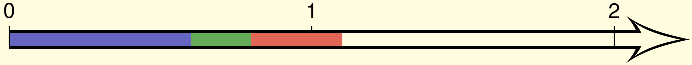
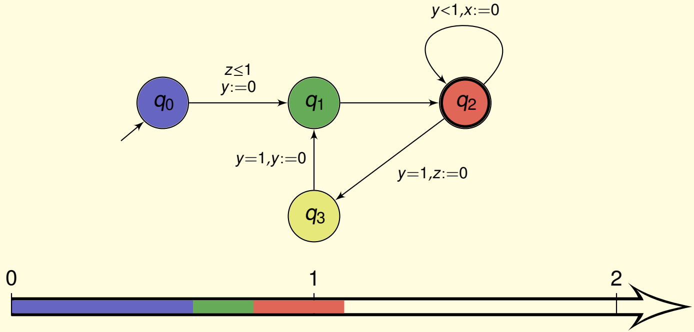




$x=$	0	0.6	0.8	1.1
$y=$	0	0	0.2	0.5
$z=$	0	0.6	0.8	1.1

Timed automata

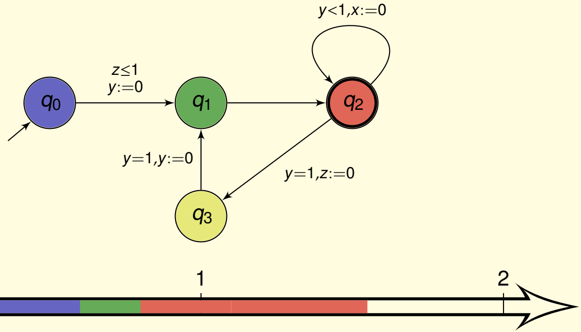
Example



$x=$	0	0.6	0.8	0
$y=$	0	0	0.2	0.5
$z=$	0	0.6	0.8	1.1

Timed automata

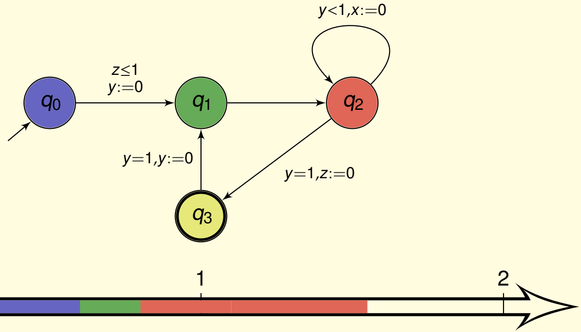
Example



$x=$	0	0.6	0.8	0	0.5
$y=$	0	0	0.2	0.5	1
$z=$	0	0.6	0.8	1.1	1.6

Timed automata

Example



$x=$	0	0.6	0.8	0	0.5
$y=$	0	0	0.2	0.5	1
$z=$	0	0.6	0.8	1.1	0

Timed automata

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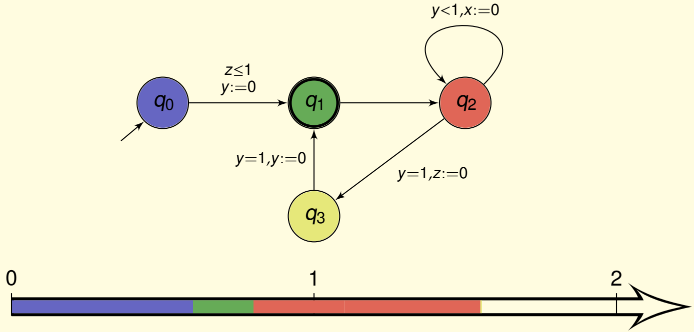
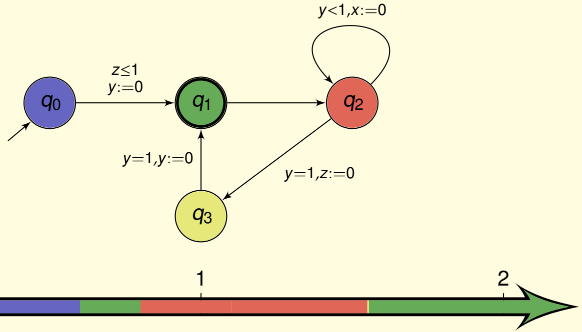


$x=$	0	0.6	0.8	0	0.5	0.5
$y=$	0	0	0.2	0.5	1	0
$z=$	0	0.6	0.8	1.1	0	0

Timed automata

Example



$x=$	0	0.6	0.8	0	0.5	0.5
$y=$	0	0	0.2	0.5	1	0
$z=$	0	0.6	0.8	1.1	0	0

Timed automata

Definition

A *timed state sequence* is a function $\pi \colon \mathbb{R}^+ \to 2^{\mathsf{AP}}$.

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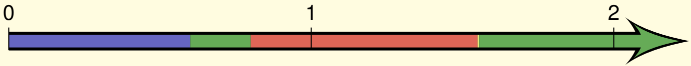

Outline of the course

1. Branching-time temporal logics
   - Complexity
   - Alternating-time Temporal Logic

2. Timed temporal logics
   - Timed models
   - Timed logics
   - Undecidability

Extending temporal logics with time

Two different ways of extending temporal logics:

- by associating intervals with modalities: those intervals (having rational bounds) indicate e.g. the moment at which an eventuality is to be fulfilled.

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$$\mathbf{G}(\texttt{call}_3 \Rightarrow \mathbf{F}_{[0,1]}\, \texttt{open}_3)$$

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$$\mathbf{A}\,\mathbf{G}(\mathbf{E}\,\mathbf{F}_{[0,3]}\, \mathtt{open}_1)$$

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- by associating intervals with modalities: those intervals (having rational bounds) indicate e.g. the moment at which an eventuality is to be fulfilled.
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Examples

$$\mathbf{G}(\mathtt{call}_3 \Rightarrow x.\mathbf{F}(\mathtt{open}_3 \wedge x \leq 1))$$

$$\mathbf{A}\,\mathbf{G}(x.\mathbf{E}\,\mathbf{F}(\mathtt{open}_1 \wedge x \leq 3))$$

Timed logics in the pointwise framework

- Syntax of MTL:

$$\mathsf{MTL} \ni \varphi ::= p \mid \neg\,\varphi \mid \varphi \vee \varphi \mid \varphi\ \mathbf{U}_I\ \varphi$$

where p ranges over AP and I is an interval with bounds in $\mathbb{Q}^+ \cup \{+\infty\}$.

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- Pointwise semantics of MTL: over $\pi = ((w_i)_i, (t_i)_i)$:
  - $\pi, i \models \varphi \,\mathbf{U}_I\, \psi$ iff there exists some $j > 0$ s.t.
    - $\pi, i + j \models \psi$,
    - $\pi, i + k \models \varphi$ for all $0 < k < j$,
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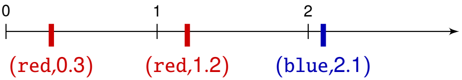

$\texttt{red}\ \mathbf{U}_{[2,3]}\ \texttt{blue}$

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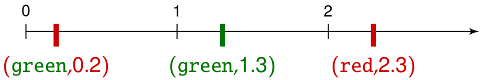

$\mathbf{F}(\texttt{green} \wedge \bot\ \mathbf{U}_{[1,1]}\ \texttt{red})$

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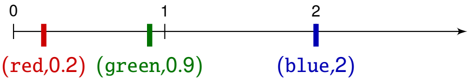

$\mathbf{F}_{[2,2]}\ \mathtt{blue}$

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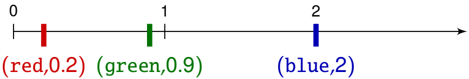

$\mathbf{F}_{[2,2]}\,\texttt{blue} \stackrel{\text{def}}{=} \mathbf{F}_{=2}\,\texttt{blue}$

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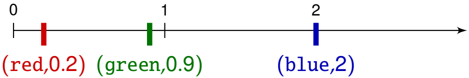

$$\mathbf{F}_{[2,2]}\,\texttt{blue} \not\equiv \mathbf{F}_{=1}\,\mathbf{F}_{=1}\,\texttt{blue}$$

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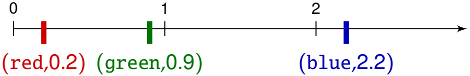

$\mathbf{F}(\mathtt{blue} \wedge \mathbf{G}^{-1}_{[-1,0]} \bot)$

Timed logics in the pointwise framework

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$$\mathsf{TPTL} \ni \varphi ::= p \mid x \sim c \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \,\mathbf{U}\, \varphi \mid x.\,\varphi$$

where p ranges over AP, x ranges over a set of formula clocks, $c \in \mathbb{Q}^+$ and $\sim \in \{<, \leq, =, \geq, >\}$.

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    - $\pi, i+j, \tau + t_{i+j} - t_i \models \psi$,
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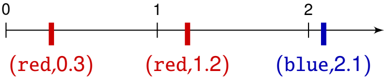

$x.(\texttt{red}\ \mathbf{U}\ (\texttt{blue} \wedge x \in [2, 3]))$

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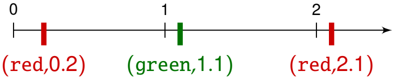

$$\mathbf{F}(\texttt{green} \wedge x.(\bot \mathbf{U} (\texttt{red} \wedge x = 1)))$$

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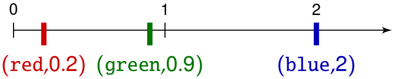

$$x.\,\mathbf{F}(\texttt{red} \wedge \mathbf{F}(\texttt{green} \wedge x \leq 1))$$

Timed logics in the continuous framework

- Syntax of MTL:

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- Continuous semantics of MTL: over $\pi : \mathbb{R}^+ \to 2^{\mathsf{AP}}$:
  - $\pi, t \models \varphi \ \mathbf{U}_I \ \psi$ iff there exists some $u > 0$ s.t.
    - $\pi, t + u \models \psi$,
    - $\pi, t + v \models \varphi$ for all $0 < v < u$,
    - $u \in I$.

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Outline of the course

1. Branching-time temporal logics
   - Complexity
   - Alternating-time Temporal Logic
2. Timed temporal logics
   - Timed models
   - Timed logics
   - Undecidability

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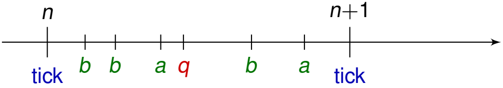

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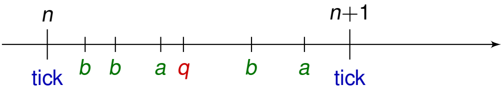

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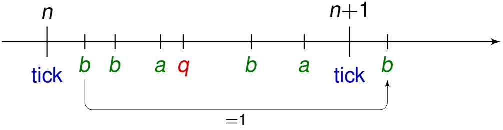

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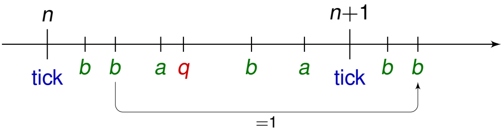

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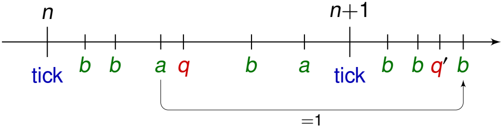

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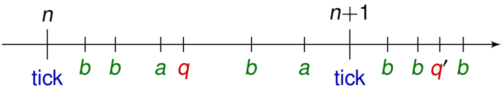
- the final state of the Turing machine is eventually reached.



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In the continuous semantics, with any MITL formula, we can associate a timed automaton that accepts exactly the same set of timed state sequences.

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In the continuous semantics, satisfiability of an MITL formula is EXPSPACE-complete.