# Probabilités discrètes et continues

## 1 Simulation of discrete distributions

### 1.1 Dice

In the early 1600s, Galileo was asked to explain the fact that, although the number of triples of integers from 1 to 6 with sum 9 is the same as the number of such triples with sum 10, when three dice are rolled, a 9 seemed to come up less often than a 10, supposedly in the experience of gamblers.

▷ **Question 1** *Write a program to simulate the roll of three dice a large number of times and keep track of the proportion of times that the sum is 9 and the proportion of times it is 10.*
Réponse: Programme simulant un lancé de trois dés un grand nombre de fois (argument $k$) et retournant la proportion de lancers où la somme des dés est 9 et celle où elle vaut 10.

```
a9 := 0
a10 := 0
for i = 1 to k do
    aux := rand(1..6) + rand(1..6) + rand(1..6)
    if aux = 9 then incrémenter a9
    else if aux = 10 then incrémenter a10
return (a9/k, a10/k)
```

**Algorithme 1** : roll($k$)

▷ **Question 2** *Were the gamblers correct ?*
Réponse: On écrit tous les triplets dans l'ordre lexicographique avec le nombre de fois qu'ils apparaissent (nombre de permutations possibles).

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | $1,2,6$ (×6) | $1,3,5$ (×6) | $1,4,4$ (×3) | $2,2,5$ (×3) | $2,3,4$ (×6) | $3,3,3$ (×**1**) |
| 10 | $1,3,6$ (×6) | $1,4,5$ (×6) | $2,2,6$ (×3) | $2,3,5$ (×6) | $2,4,4$ (×3) | $3,4,3$ (×**3**) |

On a donc 25 façons d'obtenir 9 et 27 d'obtenir 10. L'ensemble des possibles est $\Omega = [\![1,6]\!]^3$. D'où $P(9) = \frac{25}{6^3}$ et $P(10) = \frac{27}{6^3}$. Les joueurs ont donc raison.

### 1.2 Roulette

In Las Vegas, a roulette wheel has 38 slots numbered 0, 00, 1, 2, ..., 36. The 0 and 00 slots are green and half of the remaining 36 slots are red and half are black. A croupier spins the wheel and throws in an ivory ball. If you bet 1 dollar on red, you win 1 dollar if the ball stops in a red slot and otherwise you lose 1 dollar.

▷ **Question 3** *Write a program to find the total winnings for a player who makes 1000 bets on red.*
Réponse:

```
gain := 0
for i = 1 to 1000 do
    aux := rand(0..37)
    if aux < 18 then gain++ else gain--
return gain
```

**Algorithme 2** : roulette(1000)

Pour le tirage aléatoire, on a considéré :

$$\underbrace{1...17}_{\text{rouge}}\ \underbrace{18...35}_{\text{noir}}\ \underbrace{36}_{0}\ \underbrace{37}_{00}$$

Another form of bet for roulette is to bet that a specific number (say 17) will turn up. If the ball stops on your number, you get your dollar back plus 35 dollars. If not, you lose your dollar.

▷ **Question 4** *Write a program that will plot your winnings when you make 1000 plays betting each time on the number 17. What differences do you see in the graphs of your winnings on these two occasions ?*
Réponse: Le programme suivant calcule les gains d'un joueur pariant mille fois 35 dollars sur le 17.

```
gain = 0
for i = 1to1000 do
    aux := rand(0..37)
    if aux = 0 then gain += 35 else gain--
return gain
```

**Algorithme 3** : roulettebis(1000)

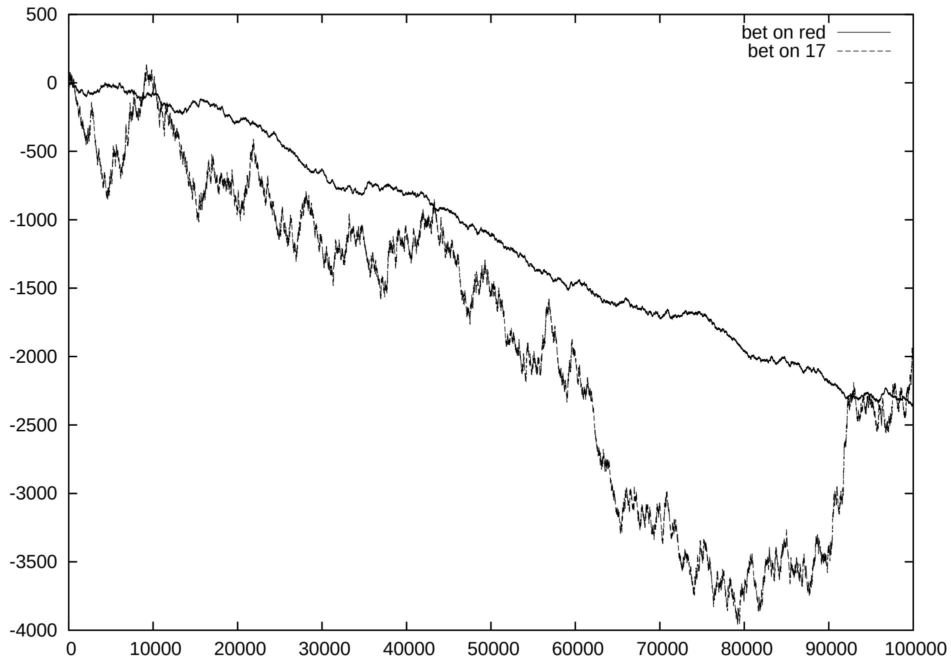


FIG. 1 – La courbe des paris sur 17 est bien plus saccadée.

Pour mieux comprendre la figure 1, calculons l'espérance les espérances correspondantes :

$$E(rouge) = 18 \times \frac{1}{38} - 20 \times \frac{1}{38} = -\frac{1}{19}$$

$$E(17) = 1 \times \frac{35}{38} - 37 \times \frac{1}{38} = -\frac{1}{19}$$

D'où $E(rouge) = E(17)$.

The Labouchere system for roulette is played as follows. Write down a list of numbers, usually 1, 2, 3, 4. Bet the sum of the first and last, 1 + 4 = 5, on red. If you win, delete the first and last numbers from your list. If you lose, add the amount that you last bet to the end of your list. Then use the new list and bet the sum of the first and last numbers (if there is only one number, bet that amount). Continue until your list becomes empty. Show that, if this happens, you win the sum, 1 + 2 + 3 + 4 = 10, of your original list.

▷ **Question 5** *Simulate this system and see if you do always stop and, hence, always win. If so, why is this not a foolproof gambling system ?*
Réponse: On peut remarquer qu'on a toujours

$$\text{gain du joueur} \ + \ \text{somme de la liste} \ = \ \text{constante}$$

En effet, ce qu'on gagne, on le retire de la liste et l'on ajoute ce qu'on perd. Ainsi lorsque la liste est vide on a bien gagné la somme de la liste initiale.

Le problème vient du fait qu'on a besoin de parier des sommes arbitrairement grandes, or en pratique il y a des limites aux mises et l'on n'a pas une quantité d'argent infinie au départ.

## 2 Discrete probability distributions

Let $\Omega = \{a, b, c\}$ be a sample space. Let $m(a) = 1/2$, $m(b) = 1/3$, and $m(c) = 1/6$.

▷ **Question 6** *Find the probabilities for all eight subsets of* $\Omega$.
Réponse: Les probabilités des huit sous-ensembles de $\Omega$ sont :

| | | |
|---|---|---|
| $m(\{a\}) = 1/2$ | $m(\{b\}) = 1/3$ | $m(\{c\}) = 1/6$ |
| $m(\{a,b\}) = m(\{a\}) + m(\{b\}) = 5/6$ | $m(\{a,c\}) = 2/3$ | $m(\{b,c\}) = 1/2$ |
| $m(\{a,b,c\}) = 1$ | $m(\emptyset) = 0$ | |

Let $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

▷ **Question 7** *Describe in words the events specified by the following subset of* $\Omega$. *Give their probability.*

1. $E$ = *HHH,HHT,HTH,HTT.*
2. $E$ = *HHH,TTT.*
3. $E$ = *HHT,HTH,THH.*
4. $E$ = *HHT,HTH,HTT,THH,THT,TTH,TTT.*

Réponse:

1. Probabilité qu'on ait face au premier jet. $P(E) = 1/2$
2. Probabilité qu'on ait trois jets identiques. $P(E) = 1/4$
3. Probabilité qu'on ait exactement une fois pile. $P(E) = 3/8$
4. Probabilité qu'on ait au moins une fois pile. $P(E) = 7/8$

Let A and B be events such that $P(A \cap B) = 1/4$, $P(\neg A) = 1/3$, and $P(B) = 1/2$.

▷ **Question 8** *What is* $P(A \cup B)$ *?*
Réponse: On fait le calcul :

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 1 - P(\neg A) + P(B) - P(A \cap B) \\ &= {}^{2}/_{3} + {}^{1}/_{2} - {}^{1}/_{4} \\ &= \frac{11}{12} \end{aligned}$$

A student must choose exactly two out of three electives : art, French, and mathematics. He chooses art with probability 5/8, French with probability 5/8, and art and French together with probability 1/4.

▷ **Question 9** *What is the probability that he chooses mathematics ? What is the probability that he chooses either art or French ?*
Réponse: Soient les événements A : «Il choisit arts», F : «Il choisit français» et M : «Il choisit mathématiques». On a $P(M) = 1 - P(A \cap F) = {}^{3}/_{4}$ et $P(A \cup F) = 1$ (il doit choisir exactement deux matières parmi les trois).

In a horse race, the odds that Romance win are listed as 2 : 3 and that Downhill win are 1 : 2.

▷ **Question 10** *What odds should be given for the event that either Romance or Downhill wins ?*
Réponse: Soient $R$ : «Romance gagne» et $D$ :«Downhill gagne». $R$ et $D$ sont disjoints donc $P(R \cup D) = {}^{2}/_{5} + {}^{1}/_{3} = {}^{11}/_{15}$ donc les chances que Romance ou Downhill gagnent sont 11 : 4.

John and Mary are taking a mathematics course. The course has only three grades : A, B, and C. The probability that John gets a B is .3. The probability that Mary gets a B is .4. The probability that neither gets an A but at least one gets a B is .1.

▷ **Question 11** *What is the probability that at least one gets a B but neither gets a C ?*
Réponse: Posons $\Omega := \{AA, AB, AC, BA, BB, BC, CA, CB, CC\}$ avec la première lettre qui correspond à la note de John et la seconde à la note de Mary. Ensuite on écrit les probabilités données par l'énoncé :
$P_1 = P(BA \cup BB \cup BC) = 0{,}3$
$P_2 = P(AB \cup BB \cup CB) = 0{,}4$
$P_3 = P(BB \cup BC \cup CB) = 0{,}1$
Tous les événements sont disjoints donc
$P_1 = P(BA) + P(BB) + P(BC)$
$P_2 = P(AB) + P(BB) + P(CB)$
$P_3 = P(BB) + P(BC) + P(CB)$
On cherche $P(AB \cup BA \cup BB) = P(AB) + P(BA) + P(BB) = P_1 + P_2 - P_3 = 0{,}6$.

A, B, and C are any three events.

▷ **Question 12** *show that* $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

Réponse:

$$\begin{aligned}
P(A\cup B\cup C) &= P(A\cup B)+P(C)-P\big((A\cup B)\cap C\big)\\
&= P(A\cup B)+P(C)-P(A\cap C\ \cup\ B\cap C)\\
&= P(A)+P(B)-P(A\cap B)+P(C)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)
\end{aligned}$$

## 3 Simulation of continuous probabilities

(La suite n'est pas corrigée).

In the spinner problem, divide the unit circumference into three arcs of length 1/2, 1/3, and 1/6.

▷ **Question 13** *Write a program to simulate the spinner experiment 1000 times*

▷ **Question 14** *Write a program to estimate the area of the circle of radius 1/2 with center at (1/2, 1/2) inside the unit square by choosing 1000 points at random and give an estimation of* $\pi$

## 4 Continuous probability densities

Suppose Mr. and Mrs. Lockhorn agree to meet at the Hanover Inn between 5 :00 and 6 :00 P.M. on Tuesday. Suppose each arrives at a time between 5 :00 and 6 :00 chosen at random with uniform probability.

▷ **Question 15** *What is the distribution function for the length of time that the first to arrive has to wait for the other ? What is the density function ?*

Suppose you choose at random a real number X from the interval [2, 10].

▷ **Question 16** *Find the density function f(x) and the probability of an event E for this experiment, where E is a subinterval [a, b] of [2, 10]. Find the probability that* $X > 5$, *that* $5 < X < 7$, *and that* $X^2 - 12X + 35 > 0$.

Suppose you throw a dart at a circular target of radius 10 inches. Assuming that you hit the target and that the coordinates of the outcomes are chosen at random,

▷ **Question 17** *Find the probability that the dart falls :*
*(a) within 2 inches of the center.*
*(b) within 2 inches of the rim.*
*(c) within the first quadrant of the target.*
*(d) within the first quadrant and within 2 inches of the rim.*