Computational Interpretation for Intuitionistic Logic in Gentzen Style

The Sequent Calculus

Logical Rules for the LJ Sequent Calculus

$$\frac{}{\Gamma, A \vdash A}\ (ax) \qquad \frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\ (cut)$$

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ (Cont)$$

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \rightarrow B \vdash C}\ (\rightarrow l) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\ (\rightarrow r)$$

Discussion

– The (Cont) rule is necessary in the cut elimination procedure.

– There are different cut-free derivations for the same sequent :

$$\cfrac{\cfrac{C, C, A \vdash B}{C, A \vdash B}}{C \vdash A \rightarrow B} \qquad \cfrac{\cfrac{C, C, A \vdash B}{C, C \vdash A \rightarrow B}}{C \vdash A \rightarrow B}$$

Is there a complete sequent calculus without contraction ?

The Sequent Calculus sc

Logical Rules for the Sequent Calculus sc

$$\frac{}{A_1, \dots, A_n \vdash A_i} \; (ax) \qquad \frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B} \; (cut)$$

$$\frac{\Gamma, A \to B \vdash A \quad \Gamma, A \to B, B \vdash C}{\Gamma, A \to B \vdash C} \; (\to l)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \; (\to r)$$

Weakening and Contraction Properties

Proposition [Derivable rules]

(W) If $\Gamma \vdash_{sc} A$, then for every type B we have $\Gamma, B \vdash_{sc} A$.

(C) If $\Gamma, C, C \vdash_{sc} A$, then $\Gamma, C \vdash_{sc} A$.

Expressive Power

Lemma

- If $\Gamma \vdash_{sc} A$, then $\Gamma \vdash_{nd} A$.
- If $\Gamma \vdash_{nd} A$, then $\Gamma \vdash_{sc} A$.

The sc-term syntax

$$
\begin{array}{rlll}
t, u ::= & x & \text{(variable)} \mid \\
 & \lambda x.t & \text{(abstraction)} \mid \\
 & u[x/t] & \text{(explicit substitution)} \mid \\
 & z \texttt{ of } t \texttt{ is } x \texttt{ in } u & \text{(generalized application)}
\end{array}
$$

A term is said to be pure if it has no explicit substitutions.

Typing Rules for Sequent Calculus sc

$$\frac{}{\Gamma, x_i : A_i \vdash x_i : A_i}\ (ax) \qquad \frac{\Gamma \vdash t : A \quad \Gamma, x : A \vdash u : B}{\Gamma \vdash u[x/t] : B}\ (cut)$$

$$\frac{\Gamma, z : A \to B \vdash t : A \quad x : B, \Gamma, z : A \to B \vdash u : C}{\Gamma, z : A \to B \vdash z \texttt{ of } t \texttt{ is } x \texttt{ in } u : C}\ (\to l)$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ (\to r)$$

We denote by $\Gamma \vdash_{sc} M : A$ derivability in the system sc and by Λ_{sc} the set of well-typed terms generated by these rules.

About type derivations

We remark that every provable sequent $\Gamma \vdash t : A$ has a unique type derivation $\Gamma \vdash_{sc} t : A$. In this case, we denote by $\mathcal{T}(t)$ the type A and by $size(t)$ the size of this derivation.

Lemma (Weakening) If $\Gamma \vdash_{sc} t : A$, then $\Gamma, x : B \vdash_{sc} t : A$. Moreover, the size of the derivation of $\Gamma, x : B \vdash_{sc} t : A$ is the same as that of $\Gamma \vdash_{sc} t : A$.

Reduction rules for sequent calculus sc

$$
\begin{array}{lcl}
x[x/t] & \to & t \\
y[x/t] & \to & y \\
(\lambda z.u)[x/t] & \to & \lambda z.u[x/t] \\
(y \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/t] & \to & y \texttt{ of } u[x/t] \texttt{ is } w \texttt{ in } v[x/t] \\
(x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/y] & \to & y \texttt{ of } u[x/y] \texttt{ is } w \texttt{ in } v[x/y] \\
(x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/\lambda z.t] & \to & v[x/\lambda z.t][w/t[z/u[x/\lambda z.t]]] \\
(x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/x' \texttt{ of } t' \texttt{ is } z \texttt{ in } t] & \to & x' \texttt{ of } t' \texttt{ is } z \texttt{ in } ((x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/t])
\end{array}
$$

We denote by $\to_{sc}$ the reduction relation generated by this set of rewrite rules.

Preservation of types

Proposition [Subject Reduction] If $\Gamma \vdash_{sc} t : A$ and $t \to_{sc} t'$, then $\Gamma \vdash_{sc} t' : A$.

Useless variables

Proposition [Void variables] Let $t, u \in \Lambda_{sc}$ and z be a variable which is not free in t. If t is pure, then $t[z/u] \to^{+}{}_{sc}\, t$.

Coherence of results

Proposition [Confluence] The reduction relation $\to_{sc}$ is confluent, i.e. if $t \to^{*}_{sc} t_1$ and $t \to^{*}_{sc} t_2$, then there is t' such that $t_1 \to^{*}_{sc} t'$ and $t_2 \to^{*}_{sc} t'$.

Cut elimination

Proposition [Cut Elimination I] For every $t = u[x/v] \in \Lambda_{sc}$ where u and v are pure terms there is a pure term $t' \in \Lambda_{sc}$ such that $t \to^{*}_{sc} t'$.

Proof : Use the *measure* : $\langle \mathcal{T}(v), size(u[x/v]) \rangle$.

Proposition [Cut Elimination II] For every $t \in \Lambda_{sc}$ there is a pure term $t' \in \Lambda_{sc}$ such that $t \to^{*}_{sc} t'$.

Proof : By induction on terms.

A cut elimination strategy

$$\begin{array}{lcl}
\mathcal{S}(x) & = & x \\
\mathcal{S}(\lambda x.u) & = & \lambda x.\mathcal{S}(u) \\
\mathcal{S}(x \texttt{ of } u \texttt{ is } y \texttt{ in } v) & = & x \texttt{ of } \mathcal{S}(u) \texttt{ is } y \texttt{ in } \mathcal{S}(v) \\
\mathcal{S}(v[x/u]) & = & \mathcal{CE}(\mathcal{S}(v)[x/\mathcal{S}(u)])
\end{array}$$

$$\begin{array}{ll}
\mathcal{CE}(x[x/u]) & = u \\
\mathcal{CE}(y[x/u]) & = y \\
\mathcal{CE}((\lambda z.t)[x/u]) & = \lambda z.\mathcal{CE}(t[x/u]) \\
\mathcal{CE}((y \texttt{ of } t \texttt{ is } w \texttt{ in } v)[x/u]) & = \\
\quad y \texttt{ of } \mathcal{CE}(t[x/u]) \texttt{ is } w \texttt{ in } \mathcal{CE}(v[x/u]) & \\
\mathcal{CE}((x \texttt{ of } t \texttt{ is } w \texttt{ in } v)[x/y]) & = \\
\quad y \texttt{ of } \mathcal{CE}(t[x/y]) \texttt{ is } w \texttt{ in } \mathcal{CE}(v[x/y]) & \\
\mathcal{CE}((x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/\lambda z.t]) & = \\
\quad \mathcal{CE}(\mathcal{CE}(v[x/\lambda z.t])[w/\mathcal{CE}(t[z/\mathcal{CE}(u[x/\lambda z.t])])]) & \\
\mathcal{CE}((x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/x' \texttt{ of } t' \texttt{ is } z \texttt{ in } t]) & = \\
\quad x' \texttt{ of } t' \texttt{ is } z \texttt{ in } \mathcal{CE}(((x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/t])) &
\end{array}$$

Strong normalization properties

Lemma If $u, v \in SN(sc)$ are *typed* terms, then $v[x/u] \in SN(sc)$.

Proof :

Use the measure : $\langle \mathcal{T}(u), \nu(u) + \nu(v), size(v[x/u]) \rangle$

where $\nu(t)$ denotes the maximal length of a reduction sequence starting at t.

Theorem [Strong Normalization] If $\Gamma \vdash_{sc} t : A$, then $t \in SN(sc)$.

Proof : By induction on terms.

Discussion (i)

Can sc be seen as a refinement of β?

Define $\mathcal{T}()$ from Λ_λ to Λ_{sc} as follows

$$\begin{array}{lcl}
\mathcal{T}(x) & = & x \\
\mathcal{T}(\lambda x.u) & = & \lambda x.\mathcal{T}(u) \\
\mathcal{T}(t\ u) & = & (x \texttt{ of } \mathcal{T}(u) \texttt{ is } z \texttt{ in } z)[x/\mathcal{T}(t)]
\end{array}$$

Is that true that $t \to_\beta u$ implies $\mathcal{T}(t) \to^*_{sc} \mathcal{T}(u)$?

Discussion (ii)

- There is no bijective correspondence between normal λ-terms and cut-free proofs.
- The calculus is not well-adapted to proof search.

The Sequent Calculus mj

Motivations

- The cut-free fragment is a natural basis for automated proof search (since it is free from permutative problems).
- The whole fragment can simulate β-reduction.
- Bijection between the cut-free fragment and the normal simply typed λ-terms.

Logical Rules for Sequent Calculus mj (I)

$$\frac{}{\Gamma; A \vdash A}\ (ax) \qquad \frac{\Gamma A; A \vdash B}{\Gamma, A; \vdash B}\ (c)$$

$$\frac{\Gamma; \vdash A \quad \Gamma; B \vdash C}{\Gamma; A \to B \vdash C}\ (\to l) \qquad \frac{\Gamma, A; \vdash B}{\Gamma; \vdash A \to B}\ (\to r)$$

Logical Rules for Sequent Calculus mj (II)

$$\frac{\Gamma; B \vdash A \quad \Gamma; A \vdash C}{\Gamma; B \vdash C}\ (cut_{r_1}) \qquad \frac{\Gamma; \vdash A \quad \Gamma; A \vdash C}{\Gamma; \vdash C}\ (cut_{r_2})$$

$$\frac{\Gamma; \vdash A \quad \Gamma, A; B \vdash C}{\Gamma; B \vdash C}\ (cut_{l_1}) \qquad \frac{\Gamma; \vdash A \quad \Gamma, A; \vdash C}{\Gamma; \vdash C}\ (cut_{l_2})$$

Alternative Formulation for Cut Rules

Let Λ denote a singleton or emptyset set of formulae.

$$\frac{\Gamma; \Lambda \vdash A \quad \Gamma; A \vdash C}{\Gamma; \Lambda \vdash C}\ (cut_r)$$

$$\frac{\Gamma; \vdash A \quad \Gamma, A; \Lambda \vdash C}{\Gamma; \Lambda \vdash C}\ (cut_l)$$

Relation between sc and mj

Lemma If $\Gamma \vdash_{sc} A$, then $\Gamma; \vdash_{mj} A$.

Proof :

– The axiom

$$\Gamma, A \vdash_{sc} A \quad (ax)$$

is translated to

$$\frac{\Gamma, A; A \vdash_{mj} A \quad (ax)}{\Gamma, A; \vdash_{mj} A}\ (C)$$

– The rule

$$\frac{\Gamma, A \vdash_{sc} B}{\Gamma \vdash_{sc} A \to B}\ (\to r)$$

is translated to

$$\frac{\Gamma, A; \vdash_{mj} B}{\Gamma; \vdash_{mj} A \to B}\ (\to r)$$

– The rule

$$\frac{\Gamma, A \to B \vdash_{sc} A \quad \Gamma, A \to B, B \vdash_{sc} C}{\Gamma, A \to B \vdash_{sc} C}\ (\to l)$$

is translated to

$$\frac{\dfrac{\dfrac{\Gamma, A \to B; \vdash_{mj} A \quad \Gamma, A \to B; B \vdash_{mj} B \quad (Ax)}{\Gamma, A \to B; A \to B \vdash_{mj} B}\ (\to l)}{\Gamma, A \to B; \vdash_{mj} B}\ (c) \qquad \Gamma, A \to B, B; \vdash_{mj} C}{\Gamma, A \to B; \vdash_{mj} C}\ (cut_l$$

– The rule

$$\frac{\Gamma \vdash_{sc} A \quad \Gamma, A \vdash_{sc} B}{\Gamma \vdash_{sc} B}\ (cut)$$

is translated to

$$\frac{\Gamma; \vdash_{mj} A \quad \Gamma, A; \vdash_{mj} B}{\Gamma; \vdash_{mj} B}\ (cut_{l_2})$$

Lemma If $\Gamma; \Lambda \vdash_{mj} A$, then $\Gamma, \Lambda \vdash_{sc} A$.

Proof :

– The axiom

$$\frac{}{\Gamma; A \vdash_{mj} A}\ (Ax)$$

is translated

$$\frac{}{\Gamma, A \vdash_{sc} A}\ (Ax)$$

– The rule

$$\frac{\Gamma, A; A \vdash_{mj} B}{\Gamma, A; \vdash_{mj} B}\ (C)$$

By i.h. we have $\Gamma, A, A \vdash_{sc} B$, then $\Gamma, A \vdash_{sc} B$ since contraction is a derivable rule.

– The rule

$$\frac{\Gamma, A; \vdash_{mj} B}{\Gamma; \vdash_{mj} A \to B}\ (\to r)$$

is translated as follows

$$\frac{\Gamma, A \vdash_{sc} B}{\Gamma \vdash_{sc} A \to B}\ (\to r)$$

– The rule

$$\frac{\Gamma; \vdash_{mj} A \quad \Gamma; B \vdash_{mj} C}{\Gamma; A \to B \vdash_{mj} C}\ (\to l)$$

is translated

$$\dfrac{\dfrac{\Gamma \vdash_{sc} A}{\Gamma, A \to B \vdash_{sc} A}\ (Weakening) \qquad \dfrac{\Gamma, B \vdash_{sc} C}{\Gamma, A \to B, B \vdash_{sc} C}\ (Weakening)}{\Gamma, A \to B \vdash_{sc} C}\ (\to l)$$

– The rule

$$\frac{\Gamma; B \vdash_{mj} A \quad \Gamma; A \vdash_{mj} C}{\Gamma; B \vdash_{mj} C}\ (cut_{r_1})$$

is translated to

$$\dfrac{\Gamma, B \vdash_{sc} A \qquad \dfrac{\Gamma, A \vdash_{sc} C}{\Gamma, B, A \vdash_{sc} C}\ (Weakening)}{\Gamma, B \vdash_{sc} C}\ (cut)$$

– The rule

$$\frac{\Gamma;\vdash_{mj} A \qquad \Gamma; A \vdash_{mj} C}{\Gamma;\vdash_{mj} C}\ (cut_{r_2})$$

is translated to

$$\frac{\Gamma \vdash_{sc} A \qquad \Gamma, A \vdash_{sc} C}{\Gamma \vdash_{sc} C}\ (cut)$$

– The rule

$$\frac{\Gamma;\vdash_{mj} A \qquad \Gamma, A; B \vdash_{mj} C}{\Gamma; B \vdash_{mj} C}\ (cut_{l_1})$$

is translated to

$$\frac{\dfrac{\Gamma \vdash_{sc} A}{\Gamma, B \vdash_{sc} A}\ (Weakening) \qquad \Gamma, A, B \vdash_{sc} C}{\Gamma, B \vdash_{sc} C}\ (cut)$$

– The rule

$$\frac{\Gamma;\vdash_{mj} A \qquad \Gamma, A;\vdash_{mj} C}{\Gamma;\vdash_{mj} C}\ (cut_{l_2})$$

is translated to

$$\frac{\Gamma \vdash_{sc} A \qquad \Gamma, A \vdash_{sc} C}{\Gamma \vdash_{sc} C}\ (cut)$$

Syntax for Sequent Calculus mj

mj-terms

$$t, u ::= (x\ l) \mid (t\ l) \mid \lambda x.t \mid t[x/u]$$

mj-lists

$$l, m ::= [\,] \mid [t :: l] \mid (l @ m) \mid l[x/t]$$

Intuition

The mj-term

$$x\ [t_1 :: t_2 :: \ldots :: t_n :: [\,]]$$

represents the (normal) λ-term

$$(\ldots((x\ t_1)\ t_2)\ \ldots)\ t_n$$

Typing Rules for Sequent Calculus mj (I)

A terme t is typable with a judgement of the form $\Gamma; \vdash_{mj} t : A$

A list l is typable with a judgement of the form $\Gamma; B \vdash_{mj} l : A$.

$$\frac{}{\Gamma; A \vdash [\,] : A}\ (ax) \qquad \frac{\Gamma, x : A; A \vdash l : B}{\Gamma, x : A; \vdash x\ l : B}\ (c)$$

$$\frac{\Gamma; \vdash t : A \qquad \Gamma; B \vdash l : C}{\Gamma; A \to B \vdash [t :: l] : C}\ (\to l) \qquad \frac{\Gamma, x : A; \vdash t : B}{\Gamma; \vdash \lambda x.t : A \to B}\ (\to r)$$

Typing Rules for Sequent Calculus mj (II)

$$\frac{\Gamma; B \vdash m : A \qquad \Gamma; A \vdash l : C}{\Gamma; B \vdash m@l : C}\ (cut_{r_1})$$

$$\frac{\Gamma; \vdash t : A \qquad \Gamma; A \vdash l : C}{\Gamma; \vdash t\ l : C}\ (cut_{r_2})$$

$$\frac{\Gamma; \vdash u : A \qquad \Gamma, x : A; B \vdash l : C}{\Gamma; B \vdash l[x/u] : C}\ (cut_{l_1})$$

$$\frac{\Gamma; \vdash u : A \qquad \Gamma, x : A; \vdash t : C}{\Gamma; \vdash t[x/u] : C}\ (cut_{l_2})$$

We denote by $\vdash_{mj}$ derivability in mj and by Λ_{mj} the set of well-typed terms and well-typed lists generated by the typing rules.

Weakeninig Property

Proposition [WP]

1. If $\Gamma; \vdash_{mj} t : A$, then for every type B we have $\Gamma, x : B; \vdash_{mj} t : A$.
2. If $\Gamma; C \vdash_{mj} l : A$, then for every type B we have $\Gamma, x : B; C \vdash_{mj} l : A$.

Reduction Rules for Sequent Calculus mj (I)

$$(B) \quad (\lambda x.u)[v :: l] \rightarrow u[x/v]\ l$$

Reduction Rules for Sequent Calculus mj (II)

$$
\begin{array}{lcl}
[\,]@l & \rightarrow & l \\
[u :: l]@m & \rightarrow & [u :: (l@m)] \\
[\,][x/v] & \rightarrow & [\,] \\
[u :: l][x/v] & \rightarrow & [u[x/v] :: l[x/v]] \\
(x\ l)\ m & \rightarrow & x\ (l@m) \\
(\lambda x.u)[\,] & \rightarrow & (\lambda x.u) \\
(y\ l)[x/v] & \rightarrow & (y\ l[x/v]) \\
(x\ l)[x/v] & \rightarrow & (v\ l[x/v]) \\
(\lambda y.t)[x/v] & \rightarrow & \lambda y.t[x/v]
\end{array}
$$

Reduction Rules for Sequent Calculus mj (III)

$$
\begin{array}{lcl}
(t@l)@m & \rightarrow & t@(l@m) \\
(l@m)[x/v] & \rightarrow & l[x/v]@m[x/v] \\
(t\ l)\ m & \rightarrow & t\ (l@m) \\
(t\ l)[x/v] & \rightarrow & (t[x/v]\ l[x/v])
\end{array}
$$

Subject Reduction

Lemma

1. If $\Gamma; \vdash_{mj} t : A$ and $t \rightarrow_{mj} t'$, then $\Gamma; \vdash_{mj} t' : A$.
2. If $\Gamma; B \vdash_{mj} l : A$ and $l \rightarrow_{mj} l'$, then $\Gamma; B \vdash_{mj} l' : A$.

Fundamental Properties

Theorem [Confluence] The reduction relation $\rightarrow_{mj}$ is confluent.

Theorem [Strong Normalization]

1. If $\Gamma; \vdash_{mj} t : A$, then $t \in SN(mj)$.
2. If $\Gamma; B \vdash_{mj} l : A$, then $l \in SN(mj)$.

Discussion

Can mj be seen as a refinement of β?

Use the alternative syntax for λ-calculus :

λ-Terms

$$T ::= (x\ L) \mid (\lambda x.T)\ L$$

λ-Lists

$$L ::= [\,] \mid [T :: L]$$

Reduction

$$(\beta) \quad (\lambda x.U)\ [T :: L] \rightarrow U\{x \leftarrow T\}\ L$$

From λ-calculus to mj-calculus

Define $\mathcal{T}()$ from λ-terms to mj-terms and $\mathcal{L}()$ from λ-lists to mj-lists as follows :

$$\begin{array}{lcl} \mathcal{T}(x\ L) & = & (x\ \mathcal{L}(L)) \\ \mathcal{T}((\lambda x.T)\ L) & = & (\lambda x.\mathcal{T}(T))\ \mathcal{L}(L) \\ & & \\ \mathcal{L}([\,]) & = & [\,] \\ \mathcal{L}([T :: L]) & = & [\mathcal{T}(T) :: \mathcal{L}(L)] \end{array}$$

Recovering λ-calculus

Theorem [β-to-mj] If $M \rightarrow_{\beta} N$, then $\mathcal{T}(M) \rightarrow^{*}_{mj} \mathcal{T}(N)$.

Exercise

Translate the λ-term $t = (\lambda x.x\ y)\ (\lambda z.z)$ into mj. Compute the mj-normal form of t.

The Sequent Calculus mj Revisited

Relating mj and sc

$$t, u ::= x \mid C_x^y(t) \mid x \texttt{ of } t \texttt{ is } y \texttt{ in } u \mid t[x/u] \mid t[\![x/u]\!]$$

Typing Rules for mj Revisited (I)

$$\frac{}{\Gamma; x : A \vdash x : A}\ (ax) \qquad \frac{\Gamma, y : A; x : A \vdash t : B}{\Gamma, x : A; \vdash C_x^y(t\{x \leftrightarrow y\}) : B}\ (c)$$

$$\frac{\Gamma; \vdash t : A \qquad \Gamma; y : B \vdash u : C}{\Gamma; z : A \to B \vdash z \texttt{ of } t \texttt{ is } y \texttt{ in } u : C}\ (\to l)$$

$$\frac{\Gamma, x : A; \vdash t : B}{\Gamma; \vdash \lambda x.t : A \to B}\ (\to r)$$

Typing Rules for mj Revisited (II)

$$\frac{\Gamma;\Lambda \vdash u : A \quad \Gamma; x : A \vdash t : B}{\Gamma;\Lambda \vdash t[\![x/u]\!] : B} \ (cut_r)$$

$$\frac{\Gamma; \vdash u : A \quad \Gamma, x : A; \Lambda \vdash t : B}{\Gamma;\Lambda \vdash t[x/u] : B} \ (cut_l)$$

Reduction Rules for mj Revisited (I)

$$(B) \quad (z \texttt{ of } t \texttt{ is } y \texttt{ in } u)[\![z/\lambda x.v]\!] \to u[\![y/v[x/t]]\!]$$

Reduction Rules for mj Revisited (II)

$t[\![z/y]\!]$	$\to$	$t\{z \leftarrow y\}$
$z[\![z/\lambda y.t]\!]$	$\to$	$\lambda y.t$
$t[\![z/z' \texttt{ of } t' \texttt{ is } y' \texttt{ in } u']\!]$	$\to$	$z' \texttt{ of } t' \texttt{ is } y' \texttt{ in } (t[\![z/u']\!])$
$t[\![z/C^x_y(u)]\!]$	$\to$	$C^x_y(t[\![z/u]\!])$
$y[x/u]$	$\to$	y
$(\lambda y.t)[x/u]$	$\to$	$\lambda y.t[x/u]$
$C^y_x(t)[z/u]$	$\to$	$C^y_x(t[z/u])$
$C^y_x(t)[x/u]$	$\to$	$t[x/u][\![y/u]\!]$
$(z \texttt{ of } t \texttt{ is } y \texttt{ in } v)[x/u]$	$\to$	$z \texttt{ of } t[x/u] \texttt{ is } y \texttt{ in } v[x/u]$

Reduction Rules for mj Revisited (III)

$$t[\![x/u[\![y/v]\!]]\!] \quad \to \quad t[\![x/u]\!][\![y/v]\!]$$

$$t[\![y/v]\!][x/u] \quad \to \quad t[x/u][\![y/v[x/u]]\!]$$

Two rules have been colapsed ! which ones ?

Properties (again)

- **(Subject Reduction)** If $\Gamma; \Lambda \vdash_{mjr} t : A$ and $t \rightarrow_{mjr} t'$, then $\Gamma; \Lambda \vdash_{mjr} t' : A$.
- **(Confluence)** The reduction relation $\rightarrow_{mjr}$ is confluent.
- **(Strong Normalization)** If $\Gamma; \Lambda \vdash_{mjr} t : A$, then $t \in SN(mjr)$.

From λ-calculus to mj revisited

Define $\mathcal{T}()$ from λ-terms to mj-terms and $\mathcal{L}()$ from variables and λ-lists to mj-lists as follows :

$$
\begin{array}{lcl}
\mathcal{T}(x\ L) & = & C^y_x(\mathcal{L}_y(L)), y \text{ fresh} \\
\mathcal{T}((\lambda x.T)\ [\,]) & = & \lambda x.\mathcal{T}(T) \\
\mathcal{T}((\lambda x.U)\ [T :: L]) & = & \mathcal{L}_z([T :: L])[\![z/\lambda x.\mathcal{T}(U)]\!], z \text{ fresh}
\end{array}
$$

$$
\begin{array}{lcl}
\mathcal{L}_y([\,]) & = & y \\
\mathcal{L}_y([T :: L]) & = & y \texttt{ of } \mathcal{T}(T) \texttt{ is } y' \texttt{ in } \mathcal{L}_{y'}(L)
\end{array}
$$

Recovering λ-calculus again

Theorem [β-to-mj] If $M \rightarrow_\beta N$, then $\mathcal{T}(M) \rightarrow^*_{mjr} \mathcal{T}(N)$.

Summary

- Calculus adapted to proof search.
- Natural interpretation of logical rules in terms of lists.
- Normal λ-terms are translated to a unique cut-free proof.
- Good properties (SR, Conf, SN, etc).