
Computational Interpretation for
Intuitionistic Logic in
Natural Deduction Style

The λ -calculus

$t, u ::= x \mid \lambda x.t \mid t u$

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Typing Rules for λ -calculus

$$\frac{}{x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i} \quad (Ax)$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \rightarrow B} \quad (\rightarrow i) \quad \frac{\Gamma \vdash t : A \rightarrow B \quad \Gamma \vdash u : A}{\Gamma \vdash t u : B} \quad (\rightarrow e)$$

We denote by $\Gamma \vdash_{\lambda} t : A$ or by $\Gamma \vdash_{nd} t : A$ the derivability relation.

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Reduction Rules for λ -calculus

$(\lambda x.t) u \rightarrow_{\beta} t\{x \leftarrow u\}$

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Properties of λ -calculus

Theorem [Confluence] If $t \rightarrow_{\beta}^* u$ and $t \rightarrow_{\beta}^* v$, then there is t' such that $u \rightarrow_{\beta}^* t'$ and $v \rightarrow_{\beta}^* t'$.

Theorem [Subject Reduction] If $\Gamma \vdash_{nd} t : A$ and $t \rightarrow_{\beta} t'$, then $\Gamma \vdash_{nd} t' : A$.

Theorem [Strong Normalization] If $\Gamma \vdash_{nd} t : A$, then $t \in SN(\beta)$, i.e., there is no infinite β -reduction sequence starting at t (every β -reduction sequence starting at t terminates).

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iff

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The λ s-calculus

$t, u ::= x \mid \lambda x.t \mid t u \mid t[x/u]$

- Free and bound variables.
- Alpha-conversion.
- Pure terms.

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Reduction rules for the λ s-calculus

$t[x/v][y/u] \equiv t[y/u][x/v]$ if $x \notin FV(u)$ & $y \notin FV(v)$

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$(\lambda x.t) u$	\rightarrow_B	$t[x/u]$	
$(t u)[x/v]$	\rightarrow	$(t[x/v] u)$	if $x \in FV(t)$ & $x \notin FV(u)$
$(t u)[x/v]$	\rightarrow	$(t u[x/v])$	if $x \notin FV(t)$ & $x \in FV(u)$
$(t u)[x/v]$	\rightarrow	$(t[x/v] u[x/v])$	if $x \in FV(t)$ & $x \in FV(u)$
$(\lambda y.t)[x/v]$	\rightarrow	$\lambda y.t[x/v]$	if $x \neq y$ & $y \notin FV(v)$
$t[x/v][y/u]$	\rightarrow	$t[y/u][x/v[y/u]]$	if $y \in FV(v)$ & $y \in FV(t)$
$t[x/v][y/u]$	\rightarrow	$t[x/v[y/u]]$	if $y \in FV(v)$ & $y \notin FV(t)$
$t[x/v]$	\rightarrow	t	if $x \notin FV(t)$
$x[x/v]$	\rightarrow	v	

We define $\mathfrak{s} = \lambda \mathfrak{s} \setminus \{B\}$.

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Computing s-normal forms

Lemma The system \mathfrak{s} is confluent and terminating and thus every term has a unique \mathfrak{s} -normal form, denoted $\mathfrak{s}\text{-nf}(t)$.

Exercise : Define an inductive function \mathfrak{s} such that $\mathfrak{s}(t) = \mathfrak{s}\text{-nf}(t)$.

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Purification

Lemma If t is a pure term, then $\mathfrak{s}(t[y/u]) = t\{y \leftarrow \mathfrak{s}(u)\}$.

Lemma The \mathfrak{s} -normal forms of $\lambda \mathfrak{s}$ -terms are pure terms.

Corollary For every term t , $\mathfrak{s}(t[y/u]) = \mathfrak{s}(t)\{y \leftarrow \mathfrak{s}(u)\}$.

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Full Composition

Exercise : Which is the minimal system $\mathcal{R} \subseteq \lambda \mathfrak{s}$ such that $t[x/u] \rightarrow_{\mathcal{R}}^* t\{x \leftarrow u\}$.

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Confluence

Lemma If $t \rightarrow_{\lambda s} t'$, then $s(t) \rightarrow_{\beta}^* s(t')$.

Theorem [Simulation] If t is a pure term and $t \rightarrow_{\beta} t'$, then $t \rightarrow_{\lambda s}^* t'$.

Theorem [Confluence] If $t \rightarrow_{\lambda s}^* t_1$ and $t \rightarrow_{\lambda s}^* t_2$, then there is t_3 s.t. $t_1 \rightarrow_{\lambda s}^* t_3$ and $t_2 \rightarrow_{\lambda s}^* t_3$.

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Normalization Properties

Définition [PSN]

Let $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Let R_1 be a relation on \mathcal{O}_1 and R_2 be a relation on \mathcal{O}_2 . We say that R_2 preserves $SN(R_1)$ iff $t \in SN(R_1)$ implies $t \in SN(R_2)$.

Proposition [PSN for λs] If $t \in SN(\beta)$, then $t \in SN(\lambda s)$.

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How PSN can be lost

Let us consider the λx -calculus :

$$\begin{aligned} (\lambda x.t) u &\rightarrow_B t[x/u] \\ (t u)[x/v] &\rightarrow (t[x/v] u[x/v]) \\ (\lambda y.t)[x/v] &\rightarrow \lambda y.t[x/v] && \text{if } x \neq y \text{ \& } y \notin FV(v) \\ t[x/v] &\rightarrow t && \text{if } x \notin FV(t) \\ x[x/v] &\rightarrow v \end{aligned}$$

Define $\lambda xc = \lambda x \cup \{Comp\}$, where

$$(Comp) \quad t[x/u][y/v] \rightarrow t[x/u[y/v]] \text{ if } y \notin FV(t)$$

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- Define $\alpha_0 = [x/(\lambda y.u)u]$ and $\alpha_{m+1} = [x/u\alpha_m]$
- Show that $u\alpha_0\alpha_{m+1} \rightarrow^* \dots u\alpha_{m+1}\alpha_{m+2}\dots$
- Show that $u\alpha_{m+1}\alpha_{n+1} \rightarrow^* \dots u\alpha_m\alpha_{n+1}\dots$
- Show that there is a λx -typable term that admits an infinite reduction sequence in the new system λxc

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$$\begin{array}{ll}
\lambda u.(\lambda x.(\lambda y.u)u)((\lambda x.u)u) & \rightarrow_B \\
\lambda u.((\lambda y.u)u)[x/(\lambda x.u)u] & \rightarrow_s^* \\
\lambda u.((\lambda y.u[x/(\lambda x.u)u])u[x/(\lambda x.u)u]) & = \\
\lambda u.((\lambda y.u\alpha_0)u\alpha_0) & \rightarrow_B \\
\lambda u.(u\alpha_0)[y/u\alpha_0] & = \\
\lambda u.(u\alpha_0\alpha_1) & \rightarrow_{\lambda xc}^* \\
\dots u\alpha_1\alpha_2 \dots & \rightarrow_{\lambda xc}^* \\
\dots u\alpha_0\alpha_2 \dots & \rightarrow_{\lambda xc}^* \\
\dots u\alpha_2\alpha_3 \dots & \rightarrow_{\lambda xc}^* \\
\dots u\alpha_1\alpha_3 \dots & \rightarrow_{\lambda xc}^* \\
\dots u\alpha_0\alpha_3 \dots & \rightarrow_{\lambda xc}^* \\
\dots u\alpha_3\alpha_4 \dots & \rightarrow_{\lambda xc}^* \\
\dots u\alpha_2\alpha_4 \dots & \rightarrow_{\lambda xc}^* \\
\dots u\alpha_1\alpha_4 \dots & \rightarrow_{\lambda xc}^* \\
\dots u\alpha_0\alpha_4 \dots & \dots
\end{array}$$

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A Special Strategy for λs -Reduction...

The *strategy* \rightsquigarrow on terms is given by an inductive definition.

$$\frac{\overline{u_n} \in \text{NF}_{\lambda s} \quad t \rightsquigarrow t'}{x\overline{u_n}t\overline{v_m} \rightsquigarrow x\overline{u_n}t'\overline{v_m}} \quad \frac{t \rightsquigarrow t'}{\lambda x.t \rightsquigarrow \lambda x.t'} \quad \frac{}{(\lambda x.t)u\overline{u_n} \rightsquigarrow t[x/u]\overline{u_n}}$$

$$\frac{u \in SN(\lambda s)}{t[x/u]\overline{v_n} \rightsquigarrow t\{x \leftarrow u\}\overline{v_n}} \quad \frac{u \notin SN(\lambda s) \quad u \rightsquigarrow u'}{t[x/u]\overline{v_n} \rightsquigarrow t[x/u']\overline{v_n}}$$

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... which is Perpetual

Theorem [Perpetuality] If $t \rightsquigarrow t'$ and $t' \in SN(\lambda s)$, then $t \in SN(\lambda s)$ ($t \notin SN(\lambda s)$ implies $t' \notin SN(\lambda s)$).

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Characterisation of $SN(\lambda\mathbf{s})$

$$\frac{t_1, \dots, t_n \in \text{ISN} \quad n \geq 0}{xt_1 \dots t_n \in \text{ISN}} \quad \frac{u[x/v]t_1 \dots t_n \in \text{ISN} \quad n \geq 0}{(\lambda x.u)vt_1 \dots t_n \in \text{ISN}}$$

$$\frac{u\{x \leftarrow v\}t_1 \dots t_n \in \text{ISN} \quad v \in \text{ISN} \quad n \geq 0}{u[x/v]t_1 \dots t_n \in \text{ISN}} \quad \frac{u \in \text{ISN}}{\lambda x.u \in \text{ISN}}$$

Lemma $SN(\lambda\mathbf{s}) = \text{ISN}$.

Proof : Uses the Perpetuality Theorem.

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Reduction Preserves Types

Theorem [Subject Reduction] If $\Gamma \vdash_{\lambda\mathbf{s}} t : A$ and $t \rightarrow_{\lambda\mathbf{s}} t'$, then $\Gamma' \vdash_{\lambda\mathbf{s}} t' : A$ for some $\Gamma' \subseteq \Gamma$.

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Typing Rules for the $\lambda\mathbf{s}$ -calculus

$$\frac{}{x : A \vdash x : A} \quad (ax) \quad \frac{\Gamma \vdash t : B}{\Gamma \setminus \{x : A\} \vdash \lambda x.t : A \rightarrow B} \quad (\rightarrow i)$$

$$\frac{\Gamma \vdash t : B \rightarrow A \quad \Delta \vdash u : B}{\Gamma \uplus \Delta \vdash t u : A} \quad (\rightarrow e)$$

$$\frac{\Gamma \vdash u : B \quad \Delta \vdash t : A}{\Gamma \uplus (\Delta \setminus \{x : B\}) \vdash t[x/u] : A} \quad (cut)$$

We denote by $\Gamma \vdash_{\lambda\mathbf{s}} t : A$ the derivability relation. We write t^A if there exists Γ s.t. $\Gamma \vdash_{\lambda\mathbf{s}} t : A$.

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Strong Normalisation

Lemma If $t^A, u^B \in SN(\lambda\mathbf{s})$, then $t\{x^B \leftarrow u\} \in SN(\lambda\mathbf{s})$.

Proof : Induction on the lexicographic 3-tuple $\langle B, \eta_{\lambda\mathbf{s}}(t), t \rangle$.

Theorem [SN for $\lambda\mathbf{s}$] If $\Gamma \vdash_{\lambda\mathbf{s}} t : A$, then $t \in SN(\lambda\mathbf{s})$.

Proof : Induction on t and previous lemma using ISN.

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Summary

The λ s-calculus is a computational interpretation of Natural Deduction plus cut enjoying the following properties :

- Full composition.
- Confluence on terms.
- Simulation of one-step β -reduction.
- Preservation of β -strong normalization.
- Strong normalization of well-typed terms.
- [Admits a translation into MELL Proof-Nets.](#)