Computational Interpretation for Intuitionistic Logic in Gentzen Style

The Sequent Calculus *sc*

Logical Rules for Sequent Calculus *sc*

$$\frac{}{A_1, \ldots, A_n \vdash A_i}\ (ax) \qquad \frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\ (cut)$$

$$\frac{\Gamma, A \to B \vdash A \quad \Gamma, A \to B, B \vdash C}{\Gamma, A \to B \vdash C}\ (\to l)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to r)$$

Expressive Power

Lemma

– If $\Gamma \vdash_{sc} A$, then $\Gamma \vdash_{nd} A$.
– If $\Gamma \vdash_{nd} A$, then $\Gamma \vdash_{sc} A$.

The sc-term syntax

$$
\begin{array}{lll}
t, u ::= & x & \text{(variable)} \mid \\
 & \lambda x.t & \text{(abstraction)} \mid \\
 & u[x/t] & \text{(explicit substitution)} \mid \\
 & z \;\texttt{of}\; t \;\texttt{is}\; x \;\texttt{in}\; u & \text{(generalized application)}
\end{array}
$$

A term is said to be pure if it has no explicit substitutions.

Typing Rules for Sequent Calculus sc

$$\frac{}{\Gamma, x_i : A_i \vdash x_i : A_i}\ (ax) \qquad \frac{\Gamma \vdash t : A \quad \Gamma, x : A \vdash u : B}{\Gamma \vdash u[x/t] : B}\ (cut)$$

$$\frac{\Gamma, z : A \to B \vdash t : A \quad x : B, \Gamma, z : A \to B \vdash u : C}{\Gamma, z : A \to B \vdash z \;\texttt{of}\; t \;\texttt{is}\; x \;\texttt{in}\; u : C}\ (\to l)$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ (\to r)$$

We denote by $\Gamma \vdash_{sc} M : A$ derivability in the system sc and by Λ_{sc} the set of well-typed terms generated by these rules.

About type derivations

We remark that every provable sequent $\Gamma \vdash t : A$ has a unique type derivation $\Gamma \vdash_{sc} t : A$. In this case, we denote by $\mathcal{T}(t)$ the type A and by $size(t)$ the size of this derivation.

Lemma (Weakening) If $\Gamma \vdash_{sc} t : A$, then $\Gamma, x : B \vdash_{sc} t : A$. Moreover, the size of the derivation of $\Gamma, x : B \vdash_{sc} t : A$ is the same as that of $\Gamma \vdash_{sc} t : A$.

Reduction rules for sequent calculus sc

$$
\begin{array}{lcl}
x[x/t] & \rightarrow & t \\
y[x/t] & \rightarrow & y \\
(\lambda z.u)[x/t] & \rightarrow & \lambda z.u[x/t] \\
(y \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/t] & \rightarrow & y \texttt{ of } u[x/t] \texttt{ is } w \texttt{ in } v[x/t] \\
(x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/y] & \rightarrow & y \texttt{ of } u[x/y] \texttt{ is } w \texttt{ in } v[x/y] \\
(x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/\lambda z.t] & \rightarrow & v[x/\lambda z.t][w/t[z/u[x/\lambda z.t]]] \\
(x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/x' \texttt{ of } t' \texttt{ is } z \texttt{ in } t] & \rightarrow & x' \texttt{ of } t' \texttt{ is } z \texttt{ in } ((x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/t])
\end{array}
$$

We denote by $\rightarrow_{sc}$ the reduction relation generated by this set of rewrite rules.

Preservation of types

Proposition [Subject Reduction] If $\Gamma \vdash_{sc} t : A$ and $t \rightarrow_{sc} t'$, then $\Gamma \vdash_{sc} t' : A$.

Useless variables

Proposition [Void variables] Let $t, u \in \Lambda_{sc}$ and z be a variable which is not free in t. If t is pure, then $t[z/u] \rightarrow^{+}_{sc} t$.

Coherence of results

Proposition [Confluence] The reduction relation $\rightarrow_{sc}$ is confluent, i.e. if $t \rightarrow^*_{sc} t_1$ and $t \rightarrow^*_{sc} t_2$, then there is t' such that $t_1 \rightarrow^*_{sc} t'$ and $t_2 \rightarrow^*_{sc} t'$.

Cut elimination

Proposition [Cut Elimination I] For every $t = u[x/v] \in \Lambda_{sc}$ where u and v are pure terms there is a pure term $t' \in \Lambda_{sc}$ such that $t \rightarrow^*_{sc} t'$.

Preuve : Use the *measure* : $\langle \mathcal{T}(v), size(u[x/v]) \rangle$.

Proposition [Cut Elimination II] For every $t \in \Lambda_{sc}$ there is a pure term $t' \in \Lambda_{sc}$ such that $t \rightarrow^*_{sc} t'$.

Preuve : By induction on terms.

A cut elimination strategy

$$
\begin{aligned}
\mathcal{S}(x) &= x \\
\mathcal{S}(\lambda x.u) &= \lambda x.\mathcal{S}(u) \\
\mathcal{S}(x \texttt{ of } u \texttt{ is } y \texttt{ in } v) &= x \texttt{ of } \mathcal{S}(u) \texttt{ is } y \texttt{ in } \mathcal{S}(v) \\
\mathcal{S}(v[x/u]) &= \mathcal{CE}(\mathcal{S}(v)[x/\mathcal{S}(u)])
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{CE}(x[x/u]) && = u \\
&\mathcal{CE}(y[x/u]) && = y \\
&\mathcal{CE}((\lambda z.t)[x/u]) && = \lambda z.\mathcal{CE}(t[x/u]) \\
&\mathcal{CE}((y \texttt{ of } t \texttt{ is } w \texttt{ in } v)[x/u]) && = \\
&\qquad y \texttt{ of } \mathcal{CE}(t[x/u]) \texttt{ is } w \texttt{ in } \mathcal{CE}(v[x/u]) && \\
&\mathcal{CE}((x \texttt{ of } t \texttt{ is } w \texttt{ in } v)[x/y]) && = \\
&\qquad y \texttt{ of } \mathcal{CE}(t[x/y]) \texttt{ is } w \texttt{ in } \mathcal{CE}(v[x/y]) && \\
&\mathcal{CE}((x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/\lambda z.t]) && = \\
&\qquad \mathcal{CE}(\mathcal{CE}(v[x/\lambda z.t])[w/\mathcal{CE}(t[z/\mathcal{CE}(u[x/\lambda z.t])])]) && \\
&\mathcal{CE}((x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/x' \texttt{ of } t' \texttt{ is } z \texttt{ in } t]) && = \\
&\qquad x' \texttt{ of } t' \texttt{ is } z \texttt{ in } \mathcal{CE}(((x \texttt{ of } u \texttt{ is } w \texttt{ in } v)[x/t])) &&
\end{aligned}
$$

Strong normalization properties

Lemma If $u, v \in SN(sc)$ are *typed* terms, then $v[x/u] \in SN(sc)$.

Preuve :

Use the measure : $\langle \mathcal{T}(u), \nu(u) + \nu(v), size(v[x/u]) \rangle$

where $\nu(t)$ denotes the maximal length of a reduction sequence starting at t.

Théorème [Strong Normalization] If $\Gamma \vdash_{sc} t : A$, then $t \in SN(sc)$.

Preuve : By induction on terms.

Discussion

Can sc be seen as a refinement of β ?

Define $\mathcal{T}()$ from Λ_λ to Λ_{sc} as follows

$$
\begin{array}{lcl}
\mathcal{T}(x) & = & x \\
\mathcal{T}(\lambda x.u) & = & \lambda x.\mathcal{T}(u) \\
\mathcal{T}(t\ u) & = & (x \ \texttt{of} \ \mathcal{T}(u) \ \texttt{is} \ z \ \texttt{in} \ z)[x/\mathcal{T}(t)]
\end{array}
$$

Is that true that $t \rightarrow_\beta u$ implies $\mathcal{T}(t) \rightarrow^*_{sc} \mathcal{T}(u)$?