The Sequent Calculus mj

Logical Rules for Sequent Calculus mj (I)

$$\frac{}{\Gamma; A \vdash A}\ (ax) \qquad \frac{\Gamma A; A \vdash B}{\Gamma, A; \vdash B}\ (c)$$

$$\frac{\Gamma; \vdash A \quad \Gamma; B \vdash C}{\Gamma; A \to B \vdash C}\ (\to l) \qquad \frac{\Gamma, A; \vdash B}{\Gamma; \vdash A \to B}\ (\to r)$$

Logical Rules for Sequent Calculus mj (II)

$$\frac{\Gamma; B \vdash A \quad \Gamma; A \vdash C}{\Gamma; B \vdash C}\ (cut_{r_1}) \qquad \frac{\Gamma; \vdash A \quad \Gamma; A \vdash C}{\Gamma; \vdash C}\ (cut_{r_2})$$

$$\frac{\Gamma; \vdash A \quad \Gamma, A; B \vdash C}{\Gamma; B \vdash C}\ (cut_{l_1}) \qquad \frac{\Gamma; \vdash A \quad \Gamma, A; \vdash C}{\Gamma; \vdash C}\ (cut_{l_2})$$

Alternative Formulation for Cut Rules

Let Λ denote a singleton or emptyset set of formulae.

$$\frac{\Gamma; \Lambda \vdash A \quad \Gamma; A \vdash C}{\Gamma; \Lambda \vdash C}\ (cut_r)$$

$$\frac{\Gamma; \vdash A \quad \Gamma, A; \Lambda \vdash C}{\Gamma; \Lambda \vdash C}\ (cut_l)$$

Relation between sc and mj

Lemma If $\Gamma \vdash_{sc} A$, then $\Gamma ; \vdash_{mj} A$.

Lemma If $\Gamma ; \Lambda \vdash_{mj} A$, then $\Gamma , \Lambda \vdash_{sc} A$.

Syntax for Sequent Calculus mj

mj-terms

$$t, u ::= (x\ l) \mid (t\ l) \mid \lambda x.t \mid t[x/u]$$

mj-lists

$$l, m ::= [\,] \mid [t :: l] \mid (l @ m) \mid l[x/t]$$

Typing Rules for Sequent Calculus mj (I)

A terme t is typable with a judgement of the form $\Gamma; \vdash_{mj} t : A$

A list l is typable with a judgement of the form $\Gamma; B \vdash_{mj} l : A$.

$$\frac{}{\Gamma; A \vdash [\,] : A}\ (ax) \qquad \frac{\Gamma, x : A; A \vdash l : B}{\Gamma, x : A; \vdash x\ l : B}\ (c)$$

$$\frac{\Gamma; \vdash t : A \quad \Gamma; B \vdash l : C}{\Gamma; A \rightarrow B \vdash [t :: l] : C}\ (\rightarrow l) \qquad \frac{\Gamma, x : A; \vdash t : B}{\Gamma; \vdash \lambda x.t : A \rightarrow B}\ (\rightarrow r)$$

Typing Rules for Sequent Calculus mj (II)

$$\frac{\Gamma; B \vdash m : A \quad \Gamma; A \vdash l : C}{\Gamma; B \vdash m@l : C}\ (cut_{r_1})$$

$$\frac{\Gamma; \vdash t : A \quad \Gamma; A \vdash l : C}{\Gamma; \vdash t\ l : C}\ (cut_{r_2})$$

$$\frac{\Gamma; \vdash u : A \quad \Gamma, x : A; B \vdash l : C}{\Gamma; B \vdash l[x/u] : C}\ (cut_{l_1})$$

$$\frac{\Gamma; \vdash u : A \quad \Gamma, x : A; \vdash t : C}{\Gamma; \vdash t[x/u] : C}\ (cut_{l_2})$$

We denote by $\vdash_{mj}$ derivability in mj and by Λ_{mj} the set of well-typed terms and well-typed lists generated by the typing rules.

Weakeninig Property

Proposition [WP]

1. If $\Gamma; \vdash_{mj} t : A$, then for every type B we have $\Gamma, x : B; \vdash_{mj} t : A$.
2. If $\Gamma; C \vdash_{mj} l : A$, then for every type B we have $\Gamma, x : B; C \vdash_{mj} l : A$.

Reduction Rules for Sequent Calculus mj (I)

$$(B) \qquad (\lambda x.u)[v :: l] \rightarrow u[x/v]\ l$$

Reduction Rules for Sequent Calculus mj (II)

$$\begin{array}{lcl}
[\,]@l & \rightarrow & l \\
[u :: l]@m & \rightarrow & [u :: (l@m)] \\
[\,][x/v] & \rightarrow & [\,] \\
[u :: l][x/v] & \rightarrow & [u[x/v] :: l[x/v]] \\
(x\ l)\ m & \rightarrow & x\ (l@m) \\
(\lambda x.u)[\,] & \rightarrow & (\lambda x.u) \\
(y\ l)[x/v] & \rightarrow & (y\ l[x/v]) \\
(x\ l)[x/v] & \rightarrow & (v\ l[x/v]) \\
(\lambda y.t)[x/v] & \rightarrow & \lambda y.t[x/v]
\end{array}$$

Reduction Rules for Sequent Calculus mj (III)

$$\begin{array}{lcl}
(t@l)@m & \to & t@(l@m) \\
(l@m)[x/v] & \to & l[x/v]@m[x/v] \\
(t\ l)\ m & \to & t\ (l@m) \\
(t\ l)[x/v] & \to & (t[x/v]\ l[x/v])
\end{array}$$

Subject Reduction

Lemma

1. If $\Gamma; \vdash_{mj} t : A$ and $t \to_{mj} t'$, then $\Gamma; \vdash_{mj} t' : A$.
2. If $\Gamma; B \vdash_{mj} l : A$ and $l \to_{mj} l'$, then $\Gamma; B \vdash_{mj} l' : A$.

Fundamental Properties

Théorème [Confluence] The reduction relation $\rightarrow_{mj}$ is confluent.

Théorème [Strong Normalization]

1. If $\Gamma; \vdash_{mj} t : A$, then $t \in SN(mj)$.
2. If $\Gamma; B \vdash_{mj} l : A$, then $l \in SN(mj)$.

Discussion

Can mj be seen as a refinement of β ?

Use the alternative syntax for λ-calculus :

λ-Terms

$$T ::= (x\ L) \mid (\lambda x.T)\ L$$

λ-Lists

$$L ::= [\,] \mid [T :: L]$$

Reduction

$$(\beta) \quad (\lambda x.U)\ [T :: L] \rightarrow\ U\{x \leftarrow T\}\ L$$

From λ-calculus to mj-calculus

Define $\mathcal{T}()$ from λ-terms to mj-terms and $\mathcal{L}()$ from λ-lists to mj-lists as follows :

$$\begin{array}{lcl} \mathcal{T}(x\ L) & = & (x\ \mathcal{L}(L)) \\ \mathcal{T}((\lambda x.T)\ L) & = & (\lambda x.\mathcal{T}(T))\ \mathcal{L}(L) \\ & & \\ \mathcal{L}([\,]) & = & [\,] \\ \mathcal{L}([T :: L]) & = & [\mathcal{T}(T) :: \mathcal{L}(L)] \end{array}$$

Recovering λ-calculus

Théorème [β-to-mj] If $M \rightarrow_\beta N$, then $\mathcal{T}(M) \rightarrow^*_{mj} \mathcal{T}(N)$.

The Sequent Calculus mj Revisited

Relating mj and sc

$$t, u ::= x \mid C_x^y(t) \mid x \texttt{ of } t \texttt{ is } y \texttt{ in } u \mid t[x/u] \mid t[\![x/u]\!]$$

Typing Rules for mj Revisited (I)

$$\frac{}{\Gamma; x : A \vdash x : A}\ (ax) \qquad \frac{\Gamma, y : A; x : A \vdash t : B}{\Gamma, x : A; \vdash C^y_x(t\{x \leftrightarrow y\}) : B}\ (c)$$

$$\frac{\Gamma; \vdash t : A \quad \Gamma; y : B \vdash u : C}{\Gamma; z : A \to B \vdash z \ \texttt{of}\ t \ \texttt{is}\ y \ \texttt{in}\ u : C}\ (\to l)$$

$$\frac{\Gamma, x : A; \vdash t : B}{\Gamma; \vdash \lambda x.t : A \to B}\ (\to r)$$

Typing Rules for mj Revisited (II)

$$\frac{\Gamma; \Lambda \vdash u : A \quad \Gamma; x : A \vdash t : B}{\Gamma; \Lambda \vdash t[\![x/u]\!] : B}\ (cut_r)$$

$$\frac{\Gamma; \vdash u : A \quad \Gamma, x : A; \Lambda \vdash t : B}{\Gamma; \Lambda \vdash t[x/u] : B}\ (cut_l)$$

Reduction Rules for mj Revisited (I)

$$(B) \qquad (z \ \texttt{of} \ t \ \texttt{is} \ y \ \texttt{in} \ u)[\![z/\lambda x.v]\!] \to u[\![y/v[x/t]]\!]$$

Reduction Rules for mj Revisited (II)

$$
\begin{array}{lcl}
t[\![z/y]\!] & \to & t\{z \leftarrow y\} \\
z[\![z/\lambda y.t]\!] & \to & \lambda y.t \\
t[\![z/z' \ \texttt{of} \ t' \ \texttt{is} \ y' \ \texttt{in} \ u']\!] & \to & z' \ \texttt{of} \ t' \ \texttt{is} \ y' \ \texttt{in} \ (t[\![z/u']\!]) \\
t[\![z/C^x_y(u)]\!] & \to & C^x_y(t[\![z/u]\!]) \\
y[x/u] & \to & y \\
(\lambda y.t)[x/u] & \to & \lambda y.t[x/u] \\
C^y_x(t)[z/u] & \to & C^y_x(t[z/u]) \\
C^y_x(t)[x/u] & \to & t[x/u][\![y/u]\!] \\
(z \ \texttt{of} \ t \ \texttt{is} \ y \ \texttt{in} \ v)[x/u] & \to & z \ \texttt{of} \ t[x/u] \ \texttt{is} \ y \ \texttt{in} \ v[x/u]
\end{array}
$$

Reduction Rules for mj Revisited (III)

$$
\begin{array}{lcl}
t[\![x/u[\![y/v]\!]]\!] & \to & t[\![x/u]\!][\![y/v]\!] \\
t[\![y/v]\!][x/u] & \to & t[x/u][\![y/v[x/u]]\!]
\end{array}
$$

Two rules have been colapsed ! which ones ?

Properties (again)

- **(Subject Reduction)** If $\Gamma; \Lambda \vdash_{mjr} t : A$ and $t \to_{mjr} t'$, then $\Gamma; \Lambda \vdash_{mjr} t' : A$.
- **(Confluence)** The reduction relation $\to_{mjr}$ is confluent.
- **(Strong Normalization)** If $\Gamma; \Lambda \vdash_{mjr} t : A$, then $t \in SN(mjr)$.

From λ-calculus to mj revisited

Define $\mathcal{T}()$ from λ-terms to mj-terms and $\mathcal{L}()$ from variables and λ-lists to mj-lists as follows :

$$\begin{array}{lcl}
\mathcal{T}(x\ L) & = & C^y_x(\mathcal{L}_y(L)), y \text{ fresh} \\
\mathcal{T}((\lambda x.T)\ [\,]) & = & \lambda x.\mathcal{T}(T) \\
\mathcal{T}((\lambda x.U)\ [T :: L]) & = & \mathcal{L}_z([T :: L])[\![z/\lambda x.\mathcal{T}(U)]\!], z \text{ fresh}
\end{array}$$

$$\begin{array}{lcl}
\mathcal{L}_y([\,]) & = & y \\
\mathcal{L}_y([T :: L]) & = & y \texttt{ of } \mathcal{T}(T) \texttt{ is } y' \texttt{ in } \mathcal{L}_{y'}(L)
\end{array}$$

Recovering λ-calculus again

Théorème [β-to-mj] If $M \rightarrow_\beta N$, then $\mathcal{T}(M) \rightarrow^*_{mjr} \mathcal{T}(N)$.

Summary

– Calculus adapted to proof search.
– Natural interpretation of logical rules in terms of lists.
– Normal λ-terms are translated to a unique cut-free proof.
– Good properties (SR, Conf, SN, etc).