Some elements for the computational interpretation of intuitionistic logic

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Contents

1 Logical Rules

1.1 Logical Rules for Natural Deduction

$$\frac{}{A_1, \ldots, A_n \vdash_{ND} A_i} \quad (Axiom)$$

$$\frac{\Gamma, A \vdash_{ND} B}{\Gamma \vdash_{ND} A \to B} \quad (\to Intro) \qquad \frac{\Gamma \vdash_{ND} A \to B \quad \Gamma \vdash_{ND} A}{\Gamma \vdash_{ND} B} \quad (\to Elim)$$

1.2 Logical Rules for Natural Deduction plus Cut

$$\frac{}{A_1, \ldots, A_n \vdash_{ND_C} A_i} \quad (Axiom) \qquad \frac{\Gamma \vdash_{ND_C} A \quad \Gamma, A \vdash_{ND_C} B}{\Gamma \vdash_{ND_C} B} \quad (Cut)$$

$$\frac{\Gamma, A \vdash_{ND_C} B}{\Gamma \vdash_{ND_C} A \to B} \quad (\to Intro) \qquad \frac{\Gamma \vdash_{ND_C} A \to B \quad \Gamma \vdash_{ND_C} A}{\Gamma \vdash_{ND_C} B} \quad (\to Elim)$$

1.3 Logical Rules for Sequent Calculus SC

$$\frac{}{A_1, \ldots, A_n \vdash_{SC} A_i} \quad (Axiom) \qquad \frac{\Gamma \vdash_{SC} A \quad \Gamma, A \vdash_{SC} B}{\Gamma \vdash_{SC} B} \quad (Cut)$$

$$\frac{\Gamma, A \to B \vdash_{SC} A \quad \Gamma, A \to B, B \vdash_{SC} C}{\Gamma, A \to B \vdash_{SC} C} \quad (\to Left) \qquad \frac{\Gamma, A \vdash_{SC} B}{\Gamma \vdash_{SC} A \to B} \quad (\to Right)$$

1.4 Logical Rules for Sequent Calculus $SC_{M(C)WC}$

$$\frac{}{A \vdash_{SC_{M(C)WC}} A} \quad (Axiom) \qquad \frac{\Gamma \vdash_{SC_{M(C)WC}} A \quad \Delta, A \vdash_{SC_{M(C)WC}} B}{\Gamma, \Delta \vdash_{SC_{M(C)WC}} B} \quad (Cut)$$

$$\frac{\Gamma \vdash_{SC_{M(C)WC}} A}{\Gamma, B \vdash_{SC_{M(C)WC}} A} \quad (Weak) \qquad \frac{\Gamma, A, A \vdash_{SC_{M(C)WC}} B}{\Gamma, A \vdash_{SC_{M(C)WC}} B} \quad (Cont)$$

$$\frac{\Gamma \vdash_{SC_{M(C)WC}} A \quad \Delta, B \vdash_{SC_{M(C)WC}} C}{\Gamma, \Delta, A \to B \vdash_{SC_{M(C)WC}} C} \quad (\to Left) \qquad \frac{\Gamma, A \vdash_{SC_{M(C)WC}} B}{\Gamma \vdash_{SC_{M(C)WC}} A \to B} \quad (\to Right)$$

1.5 Common Properties

Proposition 1.1 (Weakening) Let S be any of the previous systems. If $\Gamma \vdash_S A$ is derivable, then for every type B the sequent $\Gamma, B \vdash_S A$ is also derivable.

Proposition 1.2 (Contraction) Let S be any of the previous systems. If $\Gamma, B, B \vdash_S A$ is derivable, then $\Gamma, B \vdash_S A$ is also derivable.

2 Equivalence between logical systems

2.1 From ND to ND_C

Lemma 2.1 If $\Gamma \vdash_{ND} M : A$ is derivable, then $\Gamma \vdash_{ND_C} M : A$ is also derivable.

Proof. Trivial. ∎

2.2 From ND_C to ND

We have to show how to eliminate Cut. We will show this property via a computational interpretation of ND_C which is called $\lambda\mathtt{x}$.

2.3 From ND to SC

Lemma 2.2 If $\Gamma \vdash_{ND} M : A$ is derivable, then $\Gamma \vdash_{SC} M : A$ is also derivable.

Proof. By induction on $\Gamma \vdash_{ND} M : A$.
The axiom

$$\frac{}{A_1, \ldots, A_n \vdash_{ND} A_i} \quad (Axiom)$$

is translated into

$$\frac{}{A_1, \ldots, A_n \vdash_{SC} A_i} \quad (Axiom)$$

The rule

$$\frac{\Gamma, A \vdash_{ND} B}{\Gamma \vdash_{ND} A \to B} \quad (\to Intro)$$

is translated into

$$\frac{\Gamma, A \vdash_{SC} B}{\Gamma \vdash_{SC} A \to B} \quad (\to Right)$$

The rule

$$\frac{\Gamma \vdash_{ND} A \to B \qquad \Gamma \vdash_{ND} A}{\Gamma \vdash_{ND} B} \quad (\to Elim)$$

is translated into

$$\dfrac{\Gamma \vdash_{SC} A \to B \qquad \dfrac{\dfrac{\Gamma \vdash_{SC} A}{\Gamma, A \to B \vdash_{SC} A}(Prop\ 1.1) \qquad \Gamma, A \to B, B \vdash_{SC} B\ (Axiom)}{\Gamma, A \to B \vdash_{SC} B}(\to Left)}{\Gamma \vdash_{SC} B}(Cut)$$

∎

2.4 From SC to ND

The axiom

$$\frac{}{A_1, \ldots, A_n \vdash_{SC} A_i} \quad (Axiom)$$

is translated into

$$\frac{}{A_1, \ldots, A_n \vdash_{ND} A_i} \quad (Axiom)$$

The rule

$$\frac{\Gamma, A \vdash_{SC} B}{\Gamma \vdash_{SC} A \to B} \quad (\to Right)$$

is translated into

$$\frac{\Gamma, A \vdash_{ND} B}{\Gamma \vdash_{ND} A \to B} \quad (\to Intro)$$

The rule

$$\frac{\Gamma, A \to B \vdash_{SC} A \qquad \Gamma, A \to B, B \vdash_{SC} C}{\Gamma, A \to B \vdash_{SC} C} \quad (\to Left)$$

is translated into

$$\dfrac{\dfrac{\Gamma, A \to B, B \vdash_{ND} C}{\Gamma, A \to B \vdash_{ND} B \to C}(\to Intro) \qquad \dfrac{\Gamma, A \to B \vdash_{ND} A \to B\ (Axiom) \qquad \Gamma, A \to B \vdash_{ND} A}{\Gamma, A \to B \vdash_{ND} B}(\to Elim)}{\Gamma, A \to B \vdash_{ND} C}(\to Elim)$$

The rule

$$\frac{\Gamma \vdash_{SC} A \qquad \Gamma, A \vdash_{SC} B}{\Gamma \vdash_{SC} B} \quad (Cut)$$

is translated into

$$\dfrac{\Gamma \vdash_{ND} A \qquad \dfrac{\Gamma, A \vdash_{ND} B}{\Gamma \vdash_{ND} A \to B}(\to Intro)}{\Gamma \vdash_{ND} B} \quad (\to Elim)$$

2.5 From SC to $SC_{M(C)WC}$

The axiom

$$\frac{}{A_1, \ldots, A_n \vdash_{SC} A_i} \quad (Axiom)$$

is translated into

$$\dfrac{\dfrac{}{A_i \vdash_{SC_{M(C)WC}} A_i}\,(Axiom)}{A_1,\ldots,A_n \vdash_{SC_{M(C)WC}} A_i}\,(Prop\ 1.1)$$

The rule

$$\frac{\Gamma, A \vdash_{SC} B}{\Gamma \vdash_{SC} A \to B} \quad (\to Right)$$

is translated into

$$\frac{\Gamma, A \vdash_{SC_{M(C)WC}} B}{\Gamma \vdash_{SC_{M(C)WC}} A \to B} \quad (\to Right)$$

The rule

$$\frac{\Gamma, A \to B \vdash_{SC} A \qquad \Gamma, A \to B, B \vdash_{SC} C}{\Gamma, A \to B \vdash_{SC} C} \quad (\to Left)$$

is translated into

$$\dfrac{\dfrac{\Gamma, A \to B \vdash_{SC_{M(C)WC}} A \qquad \Gamma, A \to B, B \vdash_{SC_{M(C)WC}} C}{\Gamma, \Gamma, A \to B, A \to B \vdash_{SC_{M(C)WC}} C}\,(\to Left)}{\Gamma, A \to B \vdash_{SC_{M(C)WC}} C}\,(Prop\ 1.2)$$

The rule

$$\frac{\Gamma \vdash_{SC} A \qquad \Gamma, A \vdash_{SC} B}{\Gamma \vdash_{SC} B} \quad (Cut)$$

is translated into

$$\dfrac{\dfrac{\Gamma \vdash_{SC_{M(C)WC}} A \qquad \Gamma, A \vdash_{SC_{M(C)WC}} B}{\Gamma, \Gamma \vdash_{SC_{M(C)WC}} B}\,(Cut)}{\Gamma \vdash_{SC_{M(C)WC}} B}\,(Cont)$$

2.6 From $SC_{M(C)WC}$ to SC

The axiom

$$\frac{}{A \vdash_{SC_{M(C)WC}} A} \quad (Axiom)$$

is translated into

$$\frac{}{A \vdash_{SC} A} \quad (Axiom)$$

The rule

$$\frac{\Gamma \vdash_{SC_{M(C)WC}} A}{\Gamma, B \vdash_{SC_{M(C)WC}} A} \quad (Weak)$$

is translated into

$$\frac{\Gamma \vdash_{SC} A}{\Gamma, B \vdash_{SC} A} (Prop\ 1.1)$$

The rule

$$\frac{\Gamma, A, A \vdash_{SC_{M(C)WC}} B}{\Gamma, A \vdash_{SC_{M(C)WC}} B} \quad (Cont)$$

is translated into

$$\frac{\Gamma, A \vdash_{SC} A\ (Axiom) \qquad \Gamma, A, A \vdash_{SC} B}{\Gamma, A \vdash_{SC} B} \quad (Cut)$$

The rule

$$\frac{\Gamma, A \vdash_{SC_{M(C)WC}} B}{\Gamma \vdash_{SC_{M(C)WC}} A \to B} \quad (\to Right)$$

is translated into

$$\frac{\Gamma, A \vdash_{SC} B}{\Gamma \vdash_{SC} A \to B} \quad (\to Right)$$

The rule

$$\frac{\Gamma \vdash_{SC_{M(C)WC}} A \qquad \Delta, B \vdash_{SC_{M(C)WC}} C}{\Gamma, \Delta, A \to B \vdash_{SC_{M(C)WC}} C} \quad (\to Left)$$

is translated into

$$\frac{\dfrac{\Gamma \vdash_{SC} A}{\Gamma, \Delta, A \to B \vdash_{SC} A} (Prop\ 1.1) \qquad \dfrac{\Gamma, B \vdash_{SC} C}{\Gamma, \Delta, A \to B, B \vdash_{SC} C} (Prop\ 1.1)}{\Gamma, \Delta, A \to B \vdash_{SC} C} \quad (\to Left)$$

The rule

$$\frac{\Gamma \vdash_{SC_{M(C)WC}} A \qquad \Delta, A \vdash_{SC_{M(C)WC}} B}{\Gamma, \Delta \vdash_{SC_{M(C)WC}} B} \quad (Cut)$$

is translated into

$$\frac{\dfrac{\Gamma \vdash_{SC} A}{\Gamma, \Delta \vdash_{SC} A} (Prop\ 1.1) \qquad \dfrac{\Delta, A \vdash_{SC} B}{\Gamma, \Delta, A \vdash_{SC} B} (Prop\ 1.1)}{\Gamma, \Delta \vdash_{SC} B} \quad (Cut)$$

3 Terms for Natural Deduction

3.1 Syntax for Natural Deduction

$$M ::= x \mid \lambda x.M \mid (MM)$$

3.2 Typing Rules for Natural Deduction

$$\frac{}{x_1 : A_1, \ldots, x_n : A_n \vdash_{ND} x_i : A_i} \quad (Ax)$$

$$\frac{\Gamma, x : A \vdash_{ND} M : B}{\Gamma \vdash_{ND} \lambda x.M : A \to B} \quad (\to i) \qquad \frac{\Gamma \vdash_{ND} M : A \to B \quad \Gamma \vdash_{ND} N : A}{\Gamma \vdash_{ND} (MN) : B} \quad (\to e)$$

We denote by Λ_{ND} the set of well-typed terms of this calculus.

3.3 Reduction Rules for Natural Deduction

$$(\lambda x.M)N \longrightarrow_{\beta} M\{x \leftarrow N\}$$

3.4 Properties for Natural Deduction

Theorem 3.1 (Confluence) If $M \longrightarrow^*_{\beta} M'$ and $M \longrightarrow^*_{\beta} M''$, then there is M''' such that $M' \longrightarrow^*_{\beta} M'''$ and $M'' \longrightarrow^*_{\beta} M'''$.

Theorem 3.2 (Subject Reduction) If $\Gamma \vdash_{ND} M : A$ and $M \longrightarrow_{\beta} M'$, then $\Gamma \vdash_{ND} M' : A$.

Theorem 3.3 (Strong Normalization) If $\Gamma \vdash_{ND} M : A$, then $M \in SN_{ND}$.

4 Terms for Natural Deduction plus Cut

4.1 Syntax for Natural Deduction plus Cut

$$M ::= x \mid \lambda x.M \mid (MM) \mid M[x/M]$$

4.2 Typing Rules for Natural Deduction plus Cut

$$\frac{}{\Gamma, x : A \vdash_{\lambda\mathtt{x}} x : A} \ (Ax) \qquad \frac{\Gamma, x : B \vdash_{\lambda\mathtt{x}} M : A \quad \Gamma \vdash_{\lambda\mathtt{x}} N : B}{\Gamma \vdash_{\lambda\mathtt{x}} M[x/N] : A} \ (Cut)$$

$$\frac{\Gamma, x : A \vdash_{\lambda\mathtt{x}} M : B}{\Gamma \vdash_{\lambda\mathtt{x}} \lambda x : A.M : B \to A} \ (\to i) \qquad \frac{\Gamma \vdash_{\lambda\mathtt{x}} M : B \to A \quad \Gamma \vdash_{\lambda\mathtt{x}} N : B}{\Gamma \vdash_{\lambda\mathtt{x}} (MN) : A} \ (\to e)$$

We denote by Λ_{ND_C} or by $\Lambda_{\lambda\mathtt{x}}$ the well-typed terms of this calculus.

4.3 Reduction Rules for Natural Deduction plus Cut

$$\begin{array}{llll}
(\lambda x.M)N & \longrightarrow_B & M[x/N] & \\
(MN)[x/L] & \longrightarrow_{App} & (M[x/L]N[x/L]) & \\
(\lambda y.M)[x/L] & \longrightarrow_{Lamb} & \lambda y.M[x/L] & \text{if } x \neq y \text{ and } y \notin FV(L) \\
y[x/M] & \longrightarrow_{Var1} & y & \text{if } x \neq y \\
x[x/M] & \longrightarrow_{Var2} & M &
\end{array}$$

We denote by $\longrightarrow_{ND_C}$ or by $\longrightarrow_{\lambda\mathtt{x}}$ the reduction relation generated by the previous reduction rules.

4.4 Properties for Natural Deduction plus Cut

See the notes on $\lambda\mathtt{x}$.

5 Terms for Sequent Calculus SC

5.1 Syntax for Sequent Calculus SC

$$M ::= x \mid \lambda x.M \mid M[x/M] \mid z \texttt{ of } M \texttt{ is } x \texttt{ in } M$$

5.2 Typing Rules for Sequent Calculus SC

No weakening rule, no contraction rule.

$$\frac{}{x_1 : A_1, \ldots, x_n : A_n \vdash_{SC} x_i : A_i}\ (Ax) \qquad \frac{\Gamma \vdash_{SC} M : A \quad \Gamma, x : A \vdash_{SC} N : B}{\Gamma \vdash_{SC} N[x/M] : B}\ (Cut)$$

$$\frac{\Gamma, z : A \to B \vdash_{SC} M : A \quad x : B, \Gamma, z : A \to B \vdash_{SC} N : C}{\Gamma, z : A \to B \vdash_{SC} z \texttt{ of } M \texttt{ is } x \texttt{ in } N : C}\ (\to L)$$

$$\frac{\Gamma, x : A \vdash_{SC} M : B}{\Gamma \vdash_{SC} \lambda x.M : A \to B}\ (\to R)$$

We denote by Λ_{SC} the set of well-typed terms of this calculus. We remark that every type derivation $\Gamma \vdash_{SC} M : A$ has a unique proof. In this case, we denote by $\mathcal{T}(M)$ the type A and by $size(M)$ the size of this derivation.

Lemma 5.1 (Weakening) In both systems, if $\Gamma \vdash_{SC} M : A$, then $\Gamma, x : B \vdash_{SC} M : A$. Moreover, the size of the derivation of $\Gamma, x : B \vdash_{SC} M : A$ is the same as that of $\Gamma \vdash_{SC} M : A$.

5.3 Reduction Rules for Sequent Calculus SC

$$\begin{array}{lll}
x[x/M] & \longrightarrow_{var_1} & M \\
y[x/M] & \longrightarrow_{var_2} & y \\
(\lambda z.N)[x/M] & \longrightarrow_{lambda} & \lambda z.N[x/M] \\
(y \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/M] & \longrightarrow_{Apps} & y \texttt{ of } N[x/M] \texttt{ is } w \texttt{ in } L[x/M] \\
(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/y] & \longrightarrow_{Appv} & y \texttt{ of } N[x/y] \texttt{ is } w \texttt{ in } L[x/y] \\
(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.M] & \longrightarrow_{Appl} & L[x/\lambda z.M][w/M[z/N[x/\lambda z.M]]] \\
(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/k \texttt{ of } K \texttt{ is } z \texttt{ in } M] & \longrightarrow_{Appa} & k \texttt{ of } K \texttt{ is } z \texttt{ in } ((x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/M])
\end{array}$$

We denote by $\longrightarrow_{SC}$ the reduction relation generated by the previous set of rewrite rules.

Lemma 5.2 (Confluence) The reduction relation $\longrightarrow_{SC}$ is confluent, i.e. if $M \longrightarrow^*_{SC} M'$ and $M \longrightarrow^*_{SC} M''$, then there is M''' such that $M' \longrightarrow^*_{SC} M'''$ and $M'' \longrightarrow^*_{SC} M'''$.

Proof. The reduction system is orthogonal. ∎

Definition 5.1 (Measure of terms) The *measure* of $P[x/N]$ is given by

$$[\![P[x/N]]\!] = \langle \mathcal{T}(N), size(P[x/N]) \rangle$$

Lemma 5.3 (Cut Elimination I) Let $M = T[x/N] \in \Lambda_{SC}$ such that T and N have no explicit substitutions. Then there is $M' \in \Lambda_{SC}$ without explicit substitutions such that $M \longrightarrow^*{}_{SC} M'$.

Proof. By induction on $[\![M]\!]$.

- $M = x[x/N]$. By hypothesis N has no substitution and $M \longrightarrow_{var_1} N$ so that $M' = N$.
- $M = y[x/N]$. Since $M \longrightarrow_{var_2} y$ and y does not have explicit substitutions the statement trivially holds for $M' = y$.
- $M = (\lambda z.P)[x/N]$. We have $[\![M]\!] = \langle \mathcal{T}(N), size(M) \rangle > \langle \mathcal{T}(N), size(P[x/N]) \rangle = [\![P[x/N]]\!]$ so that by induction hypothesis there is P' such that $P[x/N] \longrightarrow^*{}_{SC} P'$ and P' has no explicit substitutions. Therefore $M \longrightarrow_{lambda} \lambda z.P[x/N] \longrightarrow^*{}_{SC} \lambda z.P'$ gives the term M' for which the statement holds.
- $M = (y \texttt{ of } P \texttt{ is } w \texttt{ in } L)[x/N]$. Since M reduces by the *Apps* rule to $y \texttt{ of } P[x/N] \texttt{ is } w \texttt{ in } L[x/N] = F$ it is sufficient to prove that $[\![M]\!] > [\![P[x/N]]\!]$ and $[\![M]\!] > [\![L[x/N]]\!]$ so that by i.h there are P' and L' without substitutions such that $P \longrightarrow^*{}_{SC} P'$ and $L \longrightarrow^*{}_{SC} L'$ and thus $M \longrightarrow^*{}_{SC} y \texttt{ of } P' \texttt{ is } w \texttt{ in } L' = M'$ and the statement holds. We have:

$$\begin{array}{ll} [\![M]\!] & = \\ [\![(y \texttt{ of } P \texttt{ is } w \texttt{ in } L)[x/N]]\!] & = \\ \langle \mathcal{T}(N), size((y \texttt{ of } P \texttt{ is } w \texttt{ in } L)[x/N]) \rangle & > \\ \langle \mathcal{T}(N), size(P[x/N]) \rangle & = \\ [\![P[x/N]]\!] & \end{array}$$

$$\begin{array}{ll} [\![M]\!] & = \\ [\![(y \texttt{ of } P \texttt{ is } w \texttt{ in } L)[x/N]]\!] & = \\ \langle \mathcal{T}(N), size((y \texttt{ of } P \texttt{ is } w \texttt{ in } L)[x/N]) \rangle & > \\ \langle \mathcal{T}(N), size(L[x/N]) \rangle & = \\ [\![L[x/N]]\!] & \end{array}$$

- $M = (x \texttt{ of } P \texttt{ is } w \texttt{ in } L)[x/y]$. Same proof as before.
- $M = (x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.P]$. Since M reduces by the *Appl* rule to $L[x/\lambda z.P][w/P[z/N[x/\lambda z.P]]] = F$ we first prove that $[\![M]\!] > [\![N[x/\lambda z.P]]\!]$ and $[\![M]\!] > [\![L[x/\lambda z.P]]\!]$,

$$\begin{array}{ll}
[\![M]\!] & = \\
[\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.P]]\!] & = \\
\langle \mathcal{T}(\lambda z.P), size(M)\rangle & > \\
\langle \mathcal{T}(\lambda z.P), size(N[x/\lambda z.P])\rangle & = \\
[\![N[x/\lambda z.P]]\!] &
\end{array}$$

$$\begin{array}{ll}
[\![M]\!] & = \\
[\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.P]]\!] & = \\
\langle \mathcal{T}(\lambda z.P), size(M)\rangle & > \\
\langle \mathcal{T}(\lambda z.P), size(L[x/\lambda z.P])\rangle & = \\
[\![L[x/\lambda z.P]]\!] &
\end{array}$$

By i.h there are N' and L' without substitutions such that $N[x/\lambda z.P] \longrightarrow^*{}_{SC} N'$ and $L[x/\lambda z.P] \longrightarrow^*{}_{SC} L'$.

We prove now that $[\![M]\!] > [\![P[z/N']]\!]$. We have $\mathcal{T}(z) = \mathcal{T}(N[x/\lambda z.P]) = \mathcal{T}(N')$ so that

$$\begin{array}{ll}
[\![M]\!] & = \\
[\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.P]]\!] & = \\
\langle \mathcal{T}(\lambda z.P), size(M)\rangle & > \\
\langle \mathcal{T}(N'), size(P[x/N'])\rangle & = \\
[\![P[x/N']]\!] &
\end{array}$$

By i.h there is P' without substitutions such that $P[x/N'] \longrightarrow^*{}_{SC} P'$.

We now remark that $\mathcal{T}(P') = P[x/N'] = \mathcal{T}(P) < \mathcal{T}(\lambda z.P)$ so that we are able to prove that $[\![M]\!] > [\![L'[w/P']]\!]$:

$$\begin{array}{ll}
[\![M]\!] & = \\
[\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.P]]\!] & = \\
\langle \mathcal{T}(\lambda z.P), size(M)\rangle & > \\
\langle \mathcal{T}(P'), size(L'[w/P'])\rangle & = \\
[\![L'[w/P']]\!] &
\end{array}$$

By i.h there is M' without substitutions such that $L'[w/P'] \longrightarrow^*{}_{SC} M'$ and therefore we obtain the result of the statement as follows:

$$\begin{array}{ll}
L[x/\lambda z.P][w/P[z/N[x/\lambda z.P]]] & \longrightarrow^*{}_{SC} \\
L'[w/P[z/N']] & \longrightarrow^*{}_{SC} \\
L'[w/P'] & \longrightarrow^*{}_{SC} \\
M' &
\end{array}$$

- $M = (x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/k \texttt{ of } K \texttt{ is } z \texttt{ in } P]$. Since M reduces by the *Appa* rule to $k \texttt{ of } K \texttt{ is } z \texttt{ in } ((x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/P]) = F$ we first prove that $[\![M]\!] > [\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/P]]\!]$. We have

$$\begin{array}{ll} [\![M]\!] & = \\ [\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/k \texttt{ of } K \texttt{ is } z \texttt{ in } P]]\!] & = \\ \langle \mathcal{T}(k \texttt{ of } K \texttt{ is } z \texttt{ in } P), size(M)\rangle & = \\ \langle \mathcal{T}(P), size(M)\rangle & > \\ \langle \mathcal{T}(P), size((x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/P])\rangle & = \\ [\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/P]]\!] & \end{array}$$

  By i.h. there is J without substitutions such that $(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/P] \longrightarrow^*{}_{SC} J$ so that $M \longrightarrow_{Appa} F \longrightarrow^*{}_{SC} k \texttt{ of } K \texttt{ is } z \texttt{ in } J = M'$, where M' is the term that verifies the statement.

∎

Lemma 5.4 (Cut Elimination II) Let $M \in \Lambda_{SC}$ be a term of type A. Then there is $M' \in \Lambda_{SC}$ without explicit substitutions such that $M \longrightarrow^*{}_{SC} M'$.

Proof. The proof is by induction on terms. The base case is when M is a variable x: then $M' = x$ and the property trivially holds since $x \longrightarrow^*{}_{SC} x$ and x has no explicit substitutions. Let us now suppose that M is not a variable.

- If $M = \lambda x.N$, then N is smaller than M, and so by induction hypothesis there exist N' without explicit substitutions such that $N \longrightarrow^*{}_{SC} N'$ and so $\lambda x.N \longrightarrow^*{}_{SC} \lambda x.N'$ give the desired term M' satisfying the statement.
- If $M = x \texttt{ of } P \texttt{ is } y \texttt{ in } N$, then P and N are smaller than M and so by induction hypothesis there are P' and N' without explicit substitutions such that $P \longrightarrow^*{}_{SC} P'$ and $N \longrightarrow^*{}_{SC} N'$. As a consequence $x \texttt{ of } P \texttt{ is } y \texttt{ in } N \longrightarrow^*{}_{SC} x \texttt{ of } P' \texttt{ is } y \texttt{ in } N'$ give the term M' without explicit substitutions satisfying the statement.
- If $M = L[x/N]$, then L and N are smaller than M and so by induction hypothesis there are L' and N' without explicit substitutions such that $P \longrightarrow^*{}_{SC} P'$ and $N \longrightarrow^*{}_{SC} N'$. As a consequence $L[x/N] \longrightarrow^*{}_{SC} L'[x/N']$. Now, we can apply Lemma **??** which gives a term M' without explicit substitutions such that $L'[x/N'] \longrightarrow^*{}_{SC} M'$, and thus M' verifies the statement since $M \longrightarrow^*{}_{SC} M'$.

∎

Lemmas 5.3 and 5.4 give a normalization strategy which is a cut elimination strategy sugested by the proofs given before.

Definition 5.2 (A Cut Elimination Strategy)

$$
\begin{array}{lcl}
\mathcal{S}(x) & = & x \\
\mathcal{S}(\lambda x.N) & = & \lambda x.\mathcal{S}(N) \\
\mathcal{S}(x \texttt{ of } N \texttt{ is } y \texttt{ in } P) & = & x \texttt{ of } \mathcal{S}(N) \texttt{ is } y \texttt{ in } \mathcal{S}(P) \\
\mathcal{S}(L[x/N]) & = & \mathcal{CE}(\mathcal{S}(L)[x/\mathcal{S}(N)]) \\
\mathcal{CE}(x[x/N]) & = & N \\
\mathcal{CE}(y[x/N]) & = & y \\
\mathcal{CE}((\lambda z.P)[x/N]) & = & \lambda z.\mathcal{CE}(P[x/N]) \\
\mathcal{CE}((y \texttt{ of } P \texttt{ is } w \texttt{ in } L)[x/N]) & = & y \texttt{ of } \mathcal{CE}(P[x/N]) \texttt{ is } w \texttt{ in } \mathcal{CE}(L[x/N]) \\
\mathcal{CE}((x \texttt{ of } P \texttt{ is } w \texttt{ in } L)[x/y]) & = & y \texttt{ of } \mathcal{CE}(P[x/N]) \texttt{ is } w \texttt{ in } \mathcal{CE}(L[x/N]) \\
\mathcal{CE}((x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.P]) & = & \mathcal{CE}(\mathcal{CE}(L[x/\lambda z.P])[w/\mathcal{CE}(P[z/\mathcal{CE}(N[x/\lambda z.P])])]) \\
\mathcal{CE}((x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/k \texttt{ of } K \texttt{ is } z \texttt{ in } M]) & = & k \texttt{ of } K \texttt{ is } z \texttt{ in } ((x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/M])
\end{array}
$$

Lemma 5.5 Let $M, N \in \Lambda_{SC}$ and z be a variable which is not free in M. If M has no explicit substitutions, then $M[z/N] \longrightarrow^+{}_{SC}\ M$.

Proof. By induction on the term M.

- If $M = y$, then $y[x/M] \longrightarrow_{var_2}\ y = M$.
- If $M = \lambda y.L$, then by α-conversion we can assume that $x \neq y$. We have $\lambda y.L[x/N] \longrightarrow_{lambda}\ \lambda y.L[x/N]$. By induction hypoyhesis $L[x/N] \longrightarrow^+{}_{SC}\ L$ and thus $\lambda y.L[x/N] \longrightarrow^+{}_{SC}\ M$.
- If $M = y \texttt{ of } P \texttt{ is } w \texttt{ in } L$, then by α-conversion we can assume that $w \neq y$, and by hypothesis we know that $y \neq x$. By applying rule *Apps* we habe $(y \texttt{ of } P \texttt{ is } w \texttt{ in } L)[x/N] \longrightarrow\ y \texttt{ of } P[x/N] \texttt{ is } w \texttt{ in } L[x/N]$. By induction hypothesis $P[x/N] \longrightarrow^+{}_{SC}\ P$ and $L[x/N] \longrightarrow^+{}_{SC}\ L$ so that $(y \texttt{ of } P \texttt{ is } w \texttt{ in } L)[x/N] \longrightarrow^+{}_{SC}\ M$.

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5.4 Subject Reduction for Sequent Calculus SC

Lemma 5.6 If $\Gamma \vdash_{SC} M : A$ and $M \longrightarrow_{SC}\ M'$, then $\Gamma \vdash_{SC} M' : A$.

Proof. By induction on $\Gamma \vdash_{SC} M : A$.

- $x[x/M] \longrightarrow\ M$.

$$\frac{\Gamma \vdash M : A \quad \Gamma, x : A \vdash x : A}{\Gamma \vdash x[x/M] : A}\,(Cut)$$

We thus have $\Gamma \vdash M : A$.

- $y[x/M] \longrightarrow\ y$

$$\dfrac{\Gamma \vdash M : A \quad \dfrac{}{\Gamma, x : A \vdash y : A}\ (Ax)}{\Gamma \vdash y[x/M]}\ (Cut)$$

$$\dfrac{}{\Gamma \vdash y : A}\ (Ax)$$

- $(\lambda z.N)[x/M] \longrightarrow\ \lambda z.N[x/M]$

$$\dfrac{\Gamma \vdash M : C \quad \dfrac{\Gamma, x : C, z : A \vdash N : B}{\Gamma, x : C \vdash \lambda z.N : A \to B}\ (R \to)}{\Gamma \vdash (\lambda z.N)[x/M] : A \to B}\ (Cut)$$

By Lemma 5.1 we have that $\Gamma \vdash M : C$ implies $\Gamma, z : A \vdash M : C$ so that

$$\dfrac{\dfrac{\Gamma, z : A \vdash M : C \quad \Gamma, x : C, z : A \vdash N : B}{\Gamma, z : A \vdash N[x/M] : B}\ (Cut)}{\Gamma \vdash \lambda z.N[x/M] : A \to B}\ (R\mathtt{x})$$

- $(y\ \mathtt{of}\ N\ \mathtt{is}\ w\ \mathtt{in}\ L)[x/M] \longrightarrow\ y\ \mathtt{of}\ N[x/M]\ \mathtt{is}\ w\ \mathtt{in}\ L[x/M]$

$$\dfrac{\Gamma \vdash M : C \quad \dfrac{\Gamma, x : C \vdash N : B \quad \Gamma, x : C, w : D \vdash L : A}{\Gamma, x : C \vdash y\ \mathtt{of}\ N\ \mathtt{is}\ w\ \mathtt{in}\ L : A}\ (L \to, y \in \Gamma)}{\Gamma \vdash (y\ \mathtt{of}\ N\ \mathtt{is}\ w\ \mathtt{in}\ L)[x/M] : A}\ (Cut)$$

$$\dfrac{\Gamma \vdash M : C \quad \Gamma, x : C \vdash N : B}{\Gamma \vdash N[x/M] : B}\ (Cut)$$

By Lemma 5.1 we have that $\Gamma \vdash M : C$ implies $\Gamma, w : D \vdash M : C$ so that

$$\dfrac{\Gamma, w : D \vdash M : C \quad \Gamma, x : C, w : D \vdash L : A}{\Gamma, w : D \vdash L[x/M] : A}\ (Cut)$$

$$\dfrac{\Gamma \vdash N[x/M] : B \quad \Gamma, w : D \vdash L[x/M] : A}{\Gamma \vdash y\ \mathtt{of}\ N[x/M]\ \mathtt{is}\ w\ \mathtt{in}\ L[x/M] : A}\ (L \to)$$

- $(x\ \mathtt{of}\ N\ \mathtt{is}\ w\ \mathtt{in}\ L)[x/y] \longrightarrow\ y\ \mathtt{of}\ N[x/y]\ \mathtt{is}\ w\ \mathtt{in}\ L[x/y]$

$$\dfrac{\dfrac{}{\Gamma \vdash y : B \to C}\ (Ax, y \in \Gamma) \quad \dfrac{\Gamma, x : B \to C \vdash N : B \quad \Gamma, x : B \to C, w : C \vdash L : A}{\Gamma, x : B \to C \vdash x\ \mathtt{of}\ N\ \mathtt{is}\ w\ \mathtt{in}\ L : A}(L \to)}{\Gamma \vdash (x\ \mathtt{of}\ N\ \mathtt{is}\ w\ \mathtt{in}\ L)[x/y] : A}\ (Cut)$$

$$\dfrac{\Gamma \vdash y : B \to C \quad \Gamma, x : B \to C \vdash N : B}{\Gamma \vdash N[x/y] : B}\ (Cut)$$

By Lemma 5.1 we have that $\Gamma \vdash y : B \to C$ implies $\Gamma, w : C \vdash y : B \to C$ so that

$$\dfrac{\Gamma, w : C \vdash y : B \to C \quad \Gamma, x : B \to C, w : C \vdash L : A}{\Gamma, w : C \vdash L[x/y] : A}\ (Cut)$$

Finally,

$$\dfrac{\Gamma \vdash N[x/y] : B \quad \Gamma, w : C \vdash L[x/y] : A}{\Gamma \vdash y\ \mathtt{of}\ N[x/y]\ \mathtt{is}\ w\ \mathtt{in}\ L[x/y] : A}\ (L \to)$$

- $(x\ \mathtt{of}\ N\ \mathtt{is}\ w\ \mathtt{in}\ L)[x/\lambda z.M] \longrightarrow\ L[x/\lambda z.M][w/M[z/N[x/\lambda z.M]]]$

$$\dfrac{\dfrac{\Gamma, z : B \vdash M : C}{\Gamma \vdash \lambda z : B.M : B \to C}\ (L \to) \quad \dfrac{\Gamma, x : B \to C \vdash N : B \quad \Gamma, x : B \to C, w : C \vdash L : A}{\Gamma, x : B \to C \vdash x\ \mathtt{of}\ N\ \mathtt{is}\ w\ \mathtt{in}\ L : A}\ (L \to)}{\Gamma \vdash (x\ \mathtt{of}\ N\ \mathtt{is}\ w\ \mathtt{in}\ L)[x/\lambda z.M] : A}\ (C$$

$$\dfrac{\Gamma \vdash \lambda z.M : B \to C \quad \Gamma, x : B \to C \vdash N : B}{\Gamma \vdash N[x/\lambda z.M] : B}\ (Cut)$$

By Lemma 5.1 we have that $\Gamma \vdash \lambda z.M : B \to C$ implies $\Gamma, w : C \vdash \lambda z.M : B \to C$ so that

$$\dfrac{\Gamma, w : C \vdash \lambda z.M : B \to C \quad \Gamma, x : B \to C, w : C \vdash L : A}{\Gamma, w : C \vdash L[x/\lambda z.M] : A}\ (Cut)$$

$$\dfrac{\Gamma \vdash N[x/\lambda z.M] : B \quad \Gamma, z : B \vdash M : C}{\Gamma \vdash M[z/N[x/\lambda z.M]] : C}\ (Cut)$$

We finally have:

$$\dfrac{\Gamma \vdash M[z/N[x/\lambda z.M]] : C \quad \Gamma, w : C \vdash L[x/\lambda z.M] : A}{\Gamma \vdash L[x/\lambda z.M][w/M[z/N[x/\lambda z.M]]] : A}\ (Cut)$$

- $(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/k \texttt{ of } K \texttt{ is } z \texttt{ in } M] \longrightarrow\ k \texttt{ of } K \texttt{ is } z \texttt{ in } (x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/M]$.

$$\dfrac{\dfrac{\Gamma \vdash K : C \quad \Gamma, z : D \vdash M : B}{\Gamma \vdash k \texttt{ of } K \texttt{ is } z \texttt{ in } M : B}\ (L \rightarrow, k \in \Gamma) \quad \Gamma, x : B \vdash x \texttt{ of } N \texttt{ is } w \texttt{ in } L : A}{\Gamma \vdash (x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/k \texttt{ of } K \texttt{ is } z \texttt{ in } M] : A}\ (Cut)$$

By Lemma 5.1 we have that $\Gamma, x : B \vdash x \texttt{ of } N \texttt{ is } w \texttt{ in } L : A$ implies $\Gamma, z : D, x : B \vdash x \texttt{ of } N \texttt{ is } w \texttt{ in } L : A$ so that

$$\dfrac{\Gamma \vdash K : C \quad \dfrac{\Gamma, z : D \vdash M : B \quad \Gamma, z : D, x : B \vdash x \texttt{ of } N \texttt{ is } w \texttt{ in } L : A}{\Gamma, z : D \vdash (x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/M] : A}\ (Cut)}{\Gamma \vdash k \texttt{ of } K \texttt{ is } z \texttt{ in } (x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/M] : A}\ (L \rightarrow)$$

∎

5.5 Strong Normalization

Definition 5.3 (Measure of strongly normalizing terms)

$$[\![P[x/N]]\!] = \langle \mathcal{T}(N), \nu(N) + \nu(P), size(P[x/N]) \rangle$$

where $\nu(M)$ denotes the maximal length of a reduction sequence starting at M.

Lemma 5.7 (Cut of strongly normalizing terms) If $N, T \in SN_{SC}$ are *typed* terms, then $M = T[x/N] \in SN_{SC}$.

Proof. In order to show that $M = T[x/N] \in SN_{SC}$ we have to show that M only reduces to strongly normalizing terms, that is, if $M \longrightarrow_{SC} M'$, then $M' \in SN_{SC}$.

1. If the reduction takes place in internal positions of M, then $M' = T'[x/N']$ with $N \longrightarrow_{SC} N'$ or $T \longrightarrow_{SC} T'$. We have $\mathcal{T}(N) = \mathcal{T}(N')$ and $\nu(N) + \nu(P) > \nu(N') + \nu(P')$ so that $M' \in SN_{SC}$ holds by induction hypothesis.

2. If the reduction takes place at the root position of M we have to consider the following cases:

   - $M = x[x/N] \longrightarrow_{var_1} N = M'$. We have $N \in SN_{SC}$ by hypothesis.
   - $M = y[x/N] \longrightarrow_{var_1} y = M'$. We trivially have $y \in SN_{SC}$.
   - $M = (\lambda z.P)[x/N] \longrightarrow_{lambda} \lambda z.P[x/N] = M'$. Since $N, (\lambda z.P) \in SN_{SC}$ by hypothesis, then in particular also $P \in SN_{SC}$. We have $\mathcal{T}(N) = \mathcal{T}(N)$, $\nu(N) + \nu(\lambda z.P) = \nu(N) + \nu(P)$ and $size(M) > size(P[x/N])$ so that $P[x/N] \in SN_{SC}$ by i.h. and then we conclude $M' \in SN_{SC}$.
   - $M = (y \texttt{ of } P \texttt{ is } w \texttt{ in } L)[x/N] \longrightarrow M'$ by *Apps* or *Appv*, then the same proof as before.

- $M = (x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.P] \longrightarrow_{Appl} L[x/\lambda z.P][w/P[z/N[x/\lambda z.P]]] = M'$. We first remark that N, L, P are all in SN_{SC} by hypothesis. Let us see that $[\![M]\!] > [\![N[x/\lambda z.P]]\!]$ and $[\![M]\!] > [\![L[x/\lambda z.P]]\!]$,

$$\begin{array}{ll} [\![M]\!] & = \\ [\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.P]]\!] & = \\ \langle \mathcal{T}(\lambda z.P), \nu(\lambda z.P) + \nu(x \texttt{ of } N \texttt{ is } w \texttt{ in } L), size(M)\rangle & > \\ \langle \mathcal{T}(\lambda z.P), \nu(\lambda z.P) + \nu(N), size(N[x/\lambda z.P])\rangle & = \\ [\![N[x/\lambda z.P]]\!] & \end{array}$$

$$\begin{array}{ll} [\![M]\!] & = \\ [\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.P]]\!] & = \\ \langle \mathcal{T}(\lambda z.P), \nu(\lambda z.P) + \nu(x \texttt{ of } N \texttt{ is } w \texttt{ in } L), size(M)\rangle & > \\ \langle \mathcal{T}(\lambda z.P), \nu(\lambda z.P) + \nu(L), size(L[x/\lambda z.P])\rangle & = \\ [\![L[x/\lambda z.P]]\!] & \end{array}$$

  By i.h we have both $N_1 = N[x/\lambda z.P], L_1 = L[x/\lambda z.P] \in SN_{SC}$. We also know that N_1, L_1 are typed terms by construction. We now prove that $[\![M]\!] > [\![P[z/N_1]]\!]$.

$$\begin{array}{ll} [\![M]\!] & = \\ [\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.P]]\!] & = \\ \langle \mathcal{T}(\lambda z.P), ..., ...\rangle & = \\ \langle \mathcal{T}(x), ..., ...\rangle & > \\ \langle \mathcal{T}(N), ..., ...\rangle & = \\ \langle \mathcal{T}(N_1), ..., ...\rangle & = \\ [\![P[z/N_1]]\!] & \end{array}$$

  By i.h we have $P_1 = P[z/N_1] \in SN_{SC}$. We also know that P_1 is a typed term by construction. We now prove that $L_1[w/P_1] \in SN_{SC}$:

$$\begin{array}{ll} [\![M]\!] & = \\ [\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/\lambda z.P]]\!] & = \\ \langle \mathcal{T}(\lambda z.P), ..., ...\rangle & = \\ \langle \mathcal{T}(x), ..., ...\rangle & > \\ \langle \mathcal{T}(P), ..., ...\rangle & = \\ \langle \mathcal{T}(P_1), ..., ...\rangle & = \\ [\![L_1[w/P_1]]\!] & \end{array}$$

  We can thus conclude that $M' = L_1[w/P_1] \in SN_{SC}$.

- $M = (x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/k \texttt{ of } K \texttt{ is } z \texttt{ in } P] \longrightarrow_{Appa}$ $k \texttt{ of } K \texttt{ is } z \texttt{ in } ((x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/P]) = M'$.

We first prove that $(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/P] \in SN_{SC}$. We have

$$\begin{array}{lr}
[\![M]\!] & = \\
[\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/k \texttt{ of } K \texttt{ is } z \texttt{ in } P]]\!] & = \\
\langle \mathcal{T}(k \texttt{ of } K \texttt{ is } z \texttt{ in } P), \nu(k \texttt{ of } K \texttt{ is } z \texttt{ in } P) + \nu(x \texttt{ of } N \texttt{ is } w \texttt{ in } L), size(M)\rangle & = \\
\langle \mathcal{T}(P), \nu(k \texttt{ of } K \texttt{ is } z \texttt{ in } P) + \nu(x \texttt{ of } N \texttt{ is } w \texttt{ in } L), size(M)\rangle & > \\
\langle \mathcal{T}(P), \nu(P) + \nu(x \texttt{ of } N \texttt{ is } w \texttt{ in } L), size((x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/P])\rangle & = \\
[\![(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/P]]\!] &
\end{array}$$

By i.h. we obtain $(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/P] \in SN_{SC}$ and we know by construction that $(x \texttt{ of } N \texttt{ is } w \texttt{ in } L)[x/P]$ is a typed term. so that M' turns out to be also in SN_{SC}.

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Theorem 5.8 (Strong Normalization) If $\Gamma \vdash_{SC} M : A$, then $M \in SN_{SC}$.

Proof. The proof is by induction on terms. The base case is when M is a variable x which is trivially strongly normalizing. Let us now suppose that M is not a variable.

- If $M = \lambda x.N$, then N is in SN_{SC} by i.h. so that we obtain $\lambda x.N \in SN_{SC}$ directly.
- If $M = x \texttt{ of } L \texttt{ is } y \texttt{ in } N$, then L and N are in SN_{SC} by i.h. so that we obtain $x \texttt{ of } L \texttt{ is } y \texttt{ in } N \in SN_{SC}$ directly.
- If $M = L[x/N]$, then L and N are in SN_{SC} by i.h. so that $M \in SN_{SC}$ by Lemma 5.7.

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