Computational Interpretation for Intuitionistic Logic in Natural Deduction Style

Computational Interpretation for Natural Deduction (Recall)

The λ-calculus

$$t, u ::= x \mid \lambda x.t \mid (t\ u)$$

Typing Rules for λ-calculus

$$\frac{}{x_1 : A_1, \ldots, x_n : A_n \vdash x_i : A_i} \quad (Ax)$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B} \quad (\to i) \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash (t\ u) : B} \quad (\to e)$$

We denote by $\Gamma \vdash_\lambda t : A$ or by $\Gamma \vdash_{nd} t : A$ the derivability relation and by Λ_{nd} or Λ_λ the set of well-typed terms.

Reduction Rules for λ-calculus

$$(\lambda x.t)\ u \rightarrow_\beta\ t\{x \leftarrow u\}$$

Properties of λ-calculus

Théorème [Confluence] If $t \rightarrow_\beta^*\ u$ and $t \rightarrow_\beta^*\ v$, then there is t' such that $u \rightarrow_\beta^*\ t'$ and $v \rightarrow_\beta^*\ t'$.

Théorème [Subject Reduction] If $\Gamma \vdash_{nd} t : A$ and $t \rightarrow_\beta\ t'$, then $\Gamma \vdash_{nd} t' : A$.

Théorème [Strong Normalization] If $\Gamma \vdash_{nd} t : A$, then $t \in SN(\beta)$, i.e., there is no infinite β-reduction sequence starting at t (every β-reduction sequence starting at t terminates).

Computational Interpretation for Natural Deduction plus Cut

The λx-calculus

$$t, u ::= x \mid \lambda x.t \mid (t\ u) \mid t[x/u]$$

Typing Rules for the λx-calculus

$$\frac{}{\Gamma, x : A \vdash x : A}\ (ax) \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : B \rightarrow A}\ (\rightarrow i)$$

$$\frac{\Gamma, x : B \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash t[x/u] : A}\ (cut)$$

$$\frac{\Gamma \vdash t : B \rightarrow A \quad \Gamma \vdash u : B}{\Gamma \vdash (t\ u) : A}\ (\rightarrow e)$$

We denote by $\Gamma \vdash_{\lambda\mathtt{x}} t : A$ or $\Gamma \vdash_{nd_c} t : A$ the derivability relation and by $\Lambda_{\lambda\mathtt{x}}$ or Λ_{nd_c} the well-typed terms.

Reduction Rules for the λx-calculus

The λx reduction rules :

$$\begin{array}{llll}
(\lambda x.t)\ u & \rightarrow_B & t[x/u] & \\
(t\ u)[x/v] & \rightarrow & (t[x/v]\ u[x/v]) & \\
(\lambda y.t)[x/v] & \rightarrow & \lambda y.t[x/v] & \text{if } x \neq y \,\&\, y \notin FV(v) \\
y[x/v] & \rightarrow & y & \text{if } x \neq y \\
x[x/v] & \rightarrow & v &
\end{array}$$

We define $\mathtt{x} = \lambda\mathtt{x} \setminus \{B\}$.

Properties of the λx-calculus

Proposition [Subject Reduction] If $\Gamma \vdash_{\lambda\mathtt{x}} t : A$ and $t \rightarrow_{\lambda\mathtt{x}} t'$, then $\Gamma \vdash_{\lambda\mathtt{x}} t' : A$.

Purification

Lemma The system $\mathtt{x}$ is confluent and terminating and thus it defines a function.

Lemma If t is a pure term, then $\mathtt{x}(t[y/u]) = \mathtt{x}(t)\{y \leftarrow \mathtt{x}(u)\}$.

Proposition [Purification] The $\mathtt{x}$-normal forms of λx-terms are pure terms.

Corollary For every term t we have $\mathtt{x}(t[y/u]) = \mathtt{x}(t)\{y \leftarrow \mathtt{x}(u)\}$.

Purification as partial cut elimination

Proposition [From $\lambda\mathtt{x}$ to λ] For every $\Gamma \vdash_{\lambda\mathtt{x}} t : A$ we have $\Gamma \vdash_{\lambda} \mathtt{x}(t) : A$.

Confluence

Lemma If $t \longrightarrow_{\lambda\mathtt{x}} t'$, then $\mathtt{x}(t) \longrightarrow^*_{\beta} \mathtt{x}(t')$.

Lemma If t is a pure term and $t \longrightarrow_{\beta} t'$, then $t \longrightarrow^*_{\lambda\mathtt{x}} t'$.

Lemma The calculus $\lambda\mathtt{x}$ is confluent.

Normalization Properties

Définition [PSN]

Let $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Let R_1 be a relation on $\mathcal{O}_1$ and R_2 be a relation on $\mathcal{O}_2$. We say that R_2 preserves $SN(R_1)$ iff $t \in SN(R_1)$ implies $t \in SN(R_2)$.

Proposition [PSN for λx] If $t \in SN(\beta)$, then $t \in SN(\lambda\mathrm{x})$.

Proposition [SN for λx] If $\Gamma \vdash_{\lambda\mathrm{x}} t : A$, then $t \in SN(\lambda\mathrm{x})$.

Proof of PSN

- Define $\mathcal{F} = \{t \in \Lambda_{\lambda\mathrm{x}} \mid \forall\ u \subseteq t$ we have $\mathtt{x}(u) \in SN_\beta\}$
- Show that $t \in \mathcal{F}$ and $t \rightarrow_{\lambda\mathrm{x}} t'$ implies $t' \in \mathcal{F}$.
- Show that $t \in \mathcal{F}$ and $t \rightarrow_{\lambda\mathrm{x}} t'$ implies $\mathcal{T}(t) >_{rpo} \mathcal{T}(t')$, where $\mathcal{T}$ is a translation of λx-terms to first-order syntax.
- Show that $t \in \mathcal{F}$ implies $t \in SN(\lambda\mathrm{x})$ using the fact that $>_{rpo}$ is a well-founded relation.
- Conclude that $t \in SN(\beta)$ implies $t \in SN(\lambda\mathrm{x})$.

How PSN can be lost

Define $\lambda\mathrm{x}c = \lambda\mathrm{x} \cup \{Comp\}$, where

$$(Comp) \quad t[x/u][y/v] \to t[x/u[y/v]] \text{ if } y \notin FV(t)$$

- Define $\alpha_0 = [x/(\lambda y.u)u]$ and $\alpha_{m+1} = [x/u\alpha_m]$
- Show that $u\alpha_0\alpha_{m+1} \to^* \ \ldots u\alpha_{m+1}\alpha_{m+2}\ldots$
- Show that $u\alpha_{m+1}\alpha_{n+1} \to^* \ \ldots u\alpha_m\alpha_{n+1}\ldots$
- Show that there is a $\lambda\mathrm{x}$-typable term that admits an infinite reduction sequence in the new system $\lambda\mathrm{x}c$

$$\begin{array}{ll}
\lambda u.(\lambda x.(\lambda y.u)u)((\lambda x.u)u) & \to_B \\
\lambda u.((\lambda y.u)u)[x/(\lambda x.u)u] & \to^*_{\mathrm{x}} \\
\lambda u.((\lambda y.u[x/(\lambda x.u)u])u[x/(\lambda x.u)u]) & = \\
\lambda u.((\lambda y.u\alpha_0)u\alpha_0) & \to_B \\
\lambda u.(u\alpha_0)[y/u\alpha_0] & = \\
\lambda u.(u\alpha_0\alpha_1) & \to^*_{\lambda\mathrm{x}c} \\
\ldots u\alpha_1\alpha_2\ldots & \to^*_{\lambda\mathrm{x}c} \\
\ldots u\alpha_0\alpha_2\ldots & \to^*_{\lambda\mathrm{x}c} \\
\ldots u\alpha_2\alpha_3\ldots & \to^*_{\lambda\mathrm{x}c} \\
\ldots u\alpha_1\alpha_3\ldots & \to^*_{\lambda\mathrm{x}c} \\
\ldots u\alpha_0\alpha_3\ldots & \to^*_{\lambda\mathrm{x}c} \\
\ldots u\alpha_3\alpha_4\ldots & \to^*_{\lambda\mathrm{x}c} \\
\ldots u\alpha_2\alpha_4\ldots & \to^*_{\lambda\mathrm{x}c} \\
\ldots u\alpha_1\alpha_4\ldots & \to^*_{\lambda\mathrm{x}c} \\
\ldots u\alpha_0\alpha_4\ldots & \ldots
\end{array}$$

Proof of SN

To show that all $\lambda\mathtt{x}$-typable terms are in $SN(\lambda\mathtt{x})$ we show that all typable terms are in $\mathcal{F}$ (then we apply the previous result).

Let t be a $\lambda\mathtt{x}$-typable term and let $u \subseteq t$.

- Show that u is $\lambda\mathtt{x}$-typable.
- Show that $\mathtt{x}(u)$ is also $\lambda\mathtt{x}$-typable, so it is λ-typable, so it is in $SN(\beta)$.
- Conclude $t \in \mathcal{F}$.

Summary

The $\lambda\mathrm{x}$-calculus is a computational interpretation of Natural Deduction plus cut enjoying the following properties :

- Confluence on terms without (meta)variables.
- Simulation of one-step β-reduction.
- Preservation of β-strong normalization.
- Strong normalization of well-typed terms.
- Does not admit extensions with composition rules.
- Is not confluent on terms with metavariables.