Logique linéaire et paradigmes logiques du calcul

Partie II : Interprétation calculatoire des séquents

MPRI

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Planning

1. Computational interpretations for intuitionistic logic
   - Natural deduction style
   - Gentzen style
2. Computational interpretations for classical logic

Logical Systems for Intuitionistic Logic

Vocabulary : context sharing and context splitting cuts

$$\frac{\Gamma \vdash A, \Lambda \qquad A, \Gamma \vdash \Lambda}{\Gamma \vdash \Lambda}\ \text{(Sharing cut)}$$

$$\frac{\Gamma \vdash A, \Pi \qquad A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Pi, \Lambda}\ \text{(Splitting cut)}$$

Vocabulary : derivation

For any logical system S we write $\Gamma \vdash_S \Delta$ to mean that the sequent $\Gamma \vdash \Delta$ is derivable in the system S.

Natural deduction versus Gentzen style

– Introduction/Elimination vs Right/Left
– Global cut elimination vs local cut elimination

A cut in a proof in natural deduction style is an introduction followed by an elimination.
A cut in a proof in sequent style is the use of a (sharing or splitting) cut rule.

Some interesting systems

Natural deduction

$$\frac{}{\Gamma, A \vdash A}\,(ax)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,(\to i) \qquad \frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}\,(\to e)$$

Notation for derivations : $\Gamma \vdash_{nd} A$

Natural deduction plus cut

$$\frac{}{\Gamma, A \vdash A}\,(ax) \qquad \frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\,(cut)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\,(\rightarrow i) \qquad \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}\,(\rightarrow e)$$

Notation for derivations : $\Gamma \vdash_{nd_c} A$

Natural deduction plus cut and structural rules

$$\frac{}{A \vdash A}\,(ax) \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,(cut)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\,(\rightarrow i) \qquad \frac{\Gamma \vdash A \rightarrow B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}\,(\rightarrow e)$$

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\,(w) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,(c)$$

Notation for derivations : $\Gamma \vdash_{nd_{cwc}} A$

Sequent calculus

$$\frac{}{A_1, \ldots, A_n \vdash A_i}\,(ax) \qquad \frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\,(cut)$$

$$\frac{\Gamma, A \rightarrow B \vdash A \quad \Gamma, A \rightarrow B, B \vdash C}{\Gamma, A \rightarrow B \vdash C}\,(\rightarrow l)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\,(\rightarrow r)$$

Notation for derivations : $\Gamma \vdash_{sc} A$

Sequent calculus with stoups

$$\frac{}{\Gamma; A \vdash A}\quad (ax) \qquad \frac{\Gamma A; A \vdash B}{\Gamma, A; \vdash B}\quad (c)$$

$$\frac{\Gamma; \vdash A \quad \Gamma; B \vdash C}{\Gamma; A \rightarrow B \vdash C}\quad (\rightarrow l) \qquad \frac{\Gamma, A; \vdash B}{\Gamma; \vdash A \rightarrow B}\quad (\rightarrow r)$$

$$\frac{\Gamma; B \vdash A \quad \Gamma; A \vdash C}{\Gamma; B \vdash C}\ (cut_{r_1}) \qquad \frac{\Gamma; \vdash A \quad \Gamma; A \vdash C}{\Gamma; \vdash C}\ (cut_{r_2})$$

$$\frac{\Gamma; \vdash A \quad \Gamma, A; B \vdash C}{\Gamma; B \vdash C}\ (cut_{l_1}) \qquad \frac{\Gamma; \vdash A \quad \Gamma, A; \vdash C}{\Gamma; \vdash C}\ (cut_{l_2})$$

Notation for derivations : $\Gamma \vdash_{mj} A$

Common (meta)properties

Proposition [Weakening] Let S be any of the previous systems. If $\Gamma \vdash_S A$, then for every type B we have $\Gamma, B \vdash_S A$.

Proposition [Contraction] Let S be any of the previous systems. If $\Gamma, B, B \vdash_S A$, then $\Gamma, B \vdash_S A$.

Systems nd and nd_c and nd_{cwc}

Lemma If $\Gamma \vdash_{nd} M : A$, then $\Gamma \vdash_{nd_c} M : A$ and $\Gamma \vdash_{nd_{cwc}} M : A$.

Lemma If $\Gamma \vdash_{nd_c} M : A$, then $\Gamma \vdash_{nd_{cwc}} M : A$.

Lemma If $\Gamma \vdash_{nd_c} M : A$ or $\Gamma \vdash_{nd_{cwc}} M : A$, then $\Gamma \vdash_{nd} M : A$.