Appendix A

Proof of completeness for *ACC*

First of all, let's be precise about the notation we will use in this Section.

Definition A.1.1 (Sub-paired-graph) *Given a paired graph $S = (E, V, C)$, we say that $S' = (E', V', C')$ is a* sub-paired-graph *of S if*

- it is a subgraph of S (i.e. $E' \subseteq E$ and $V' \subseteq V$)
- C' is C restricted to E' (i.e. the only paired edges of S' are those edges that are already paired in S

Definition A.1.2 (Free pair) *A pair $c = \{a, b\}$ is called* free *if its basis is of degree 2 (i.e. if a, b are the only edges attached to the basis of c.*

Definition A.1.3 (Path, path composition) *A (finite) path α in a graph S is a (finite) sequence e_i of different edges of S s.t. e_n shares one node with e_{n+1}, different from the node shared with e_{n-1} in case $n > 1$. If β is a path that extends α, we note $\beta :: \alpha$ their composition.*

Definition A.1.4 (Pseudo-realizable path) *A path α in a paired graph S is* pseudo-realizable *(short,* p.r.*), if it contains only free pairs.*

Definition A.1.5 (Attractive Path) *A path α in a paired graph S is* attractive *if from every node u of S that does not belong to α there are at least two pseudo-realizable paths to α that are* distinct *in u (i.e. do not use the same edge when leaving u). We can restrict ourselves to paths that intersect α only at the end (i.e. they share just the last node with α).*

Actually, in the proof of the theorem, we will use a more relaxed condition than *ACC*.

Definition A.1.6 (Weak *ACC*) *A paired graph S satisfies* Weak-ACC *if all the graphs in $GC(S)$ are acyclic.*

This more relaxed condition is preserved by subgraphs.

Lemma A.1.7 (Sub-paired-graphs) *If S satisfies* Weak-ACC*, then any sub-paired-graph S' of S satisfies* Weak-ACC.

Proof. Notice the only non-legal configurations in a correction-graph of a paired graph S is the presence of both edges of a pair. Now, if a path α is in a graph of $GC(S')$, then it cannot contain both edges of a pair c of S'. But now, by definition of sub-paired-graph, if S' contains both edges of a pair c of S, then c is a pair in S' too, so α cannot contain both edges of a pair c of S. This means that any cycle γ in a graph of $GC(S')$ would be a cycle also in some graph of $GC(S)$, hence the statement of the lemma follows. □

Theorem A.1.8 (Paired graphs with at least one pair are split (3.2.10)) *Let S be a* finite *paired graph that satisfies* Weak-ACC. *If S contains at least one pair then it is* split.

Proof. We will show that in any paired graph S s.t.

- (H1) all graphs in $GC(S)$ are acyclic,
- (H2) with a non-empty set of pairs $\mathcal{C}(S)$,
- (H3) and with no splitting pair,

it is possible to find an infinite *strictly increasing* sequence S_n of sub-paired-graphs of S. By contradiction, the theorem will follow.
Actually, we will need to build also a sequence D_n of attractive pseudo-realizable cycles in each of the S_n. Let's remark now that in the proof we will always deal with *elementary cycles*, i.e. cyclic paths that do not contain already a cycle.
So, the inductive hypothesis we will use in the construction of the increasing sequence of sub-paired-graphs of S is the following:

- (Hi1) S_n is a sub-paired-graph of S,
- (Hi2) S_n has an attractive pseudo-realizable cycle D_n

Here comes the inductive construction.

- **Base Case**

  First, we notice that in any paired graph S satisfying H1, H2 and H3 we can actually find a cycle. Otherwise, any pair c is clearly a splitting pair (if there is no cycle in S, then it is actually a tree, and by removing the two edges of any pair c, we clearly disconnect the base from the other nodes of the pair). Take then an *elementary* cycle as the first sub-paired-graph S_0 of the sequence, and set $D_0 = S_0$.

  Notice that D_0 is trivially pseudo-realizable and attractive in S_0, since it coincides with S_0.

- **Inductive Case**

  Suppose now we are given S_n sub-paired-graph of S and D_n an elementary cycle that is pseudo-realizable and attractive in S_n.

  - **Choosing S_{n+1}.**
    First, notice that D_n must necessarily contain a pair c. Otherwise, there is a graph in $GC(S_n)$ that contains D_n, i.e. a cycle, that is impossible since S_n satisfies *Weak-ACC* by Lemma A.1.7. Furthermore, c is free in S_n because D_n is pseudo-realizable.

Let's call x the base of c, and x_1, x_2 the other nodes of c. Now, c is not a splitting pair of S, by H3. This means there is a path in S connecting x with x_1 or x_2 not using the edges of c.

Since c is *free* in S_n, such a path must necessarily use an edge e of S that does not belong to S_n. Let's call α the segment of this path that lies entirely in $S \setminus S_n$, but for x and the first node z of the path that belongs to S_n.

Let S_{n+1} be S_n augmented with this path α, considered as a sub-paired-graph of S, so that Hi1 is trivially satisfied by S_{n+1}; notice that

- S_{n+1} is strictly bigger than S_n, as at least the edge e was not in S_n
- each new pair in S_{n+1} has at least an edge in α
- the pairs in S_n that can happen to be no longer *free* have x or z as basis.

– **Building D_{n+1}.**

Now, we need to build D_{n+1}, pseudo-realizable and attractive in S_{n+1}. We will use α and D_n for this purpose. Figure A.1 will help in following the construction.

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Figure A.1: The construction of D_{n+1}

There are two cases: either z belongs to D_n or not. We will assume first that z is not in D_n.

By Hi2, D_n is attractive, so there are two paths δ_1 and δ_2 from z to D_n, distinct in z and intersecting D_n only in one node.

Let's note these nodes z_1 and z_2, respectively. Now, z_1 (resp. z_2) cannot be x, as, by construction, in S_n, x has degree 2, while z_1 (resp. z_2) have degree 3. So, one of the following paths is pseudo-realizable:

$$\alpha :: \delta_1,\ \alpha :: \delta_2$$

Indeed, δ_1 and δ_2 are pr. in S_n, so they are pseudo-realizable in S_{n+1} as they use edges that were already in S_n, and the only possible pairs and edges we add in S_{n+1} can only modify the status of a pair based in z, which is not part of these paths. Also, α is p.r. by construction. The intersection of α and δ_1 (resp. δ_2) is z. Since δ_1 and δ_2 are distinct in z, $\alpha :: \delta_1$ and $\alpha :: \delta_2$ cannot both contain a new pair in z that was not in S_n.

So, one of the two junctions is p.r.; say it is $\alpha :: \delta_1$.

Call now β_1 and β_2 the two strict sub-paths of D_n connecting z_1 to x. One of the following cycles is p.r.:

$$\alpha :: \delta_1 :: \beta_1, \alpha :: \delta_1 :: \beta_2$$

The intersection of the paths $a :: \delta_1$ and β_1 (resp. β_2) is $\{z_1, x\}$. Furthermore, β_1 and β_2 are p.r. in S_{n+1} since they are p.r. in S_n and all their pairs are still free (remember that only the degree of z and x is affected by the construction).

Similarly as above, one shows that the junction in z_1 is correct (as β_1 and β_2 are distinct in z_1).

Since the only pair with base x belongs to D_n, the junction in x is correct in both cases. Suppose now that $\alpha :: \delta_1 :: \beta_1$ is p.r.: this will be our D_{n+1}.

In case $z \in D_n$, the same proceed applies, taking z in place of z_1.

– D_{n+1} **is attractive.** Now we need to check that D_{n+1} is attractive, i.e., for each u in S_{n+1} not in D_{n+1}, we need to show two pseudo-realizable paths distinct in u connecting it to D_{n+1} (again, we can consider that such paths intersect D_{n+1} only in the final node).

Since u is not in D_{n+1}, it is necessarily in S_n (as α, that contains the only new nodes w.r.t. S_n, is entirely in D_{n+1}).

If it is in D_n, since x and z_1 are in the intersection of the two cycles, the two strict sub-paths of D_n connecting u to x and z_1 are the required p.r. paths.

If it is not in D_n, there are two p.r. paths in S_n, ϵ_1 and ϵ_2, distinct in u, connecting it to D_n (by Hi2 for S_n).

Take ϵ_1. If it does not intersect D_{n+1}, it is easy to see that one can extend it in a pseudo-realizable path to x or z_1, giving the needed path. Otherwise, it intersects D_{n+1}, and the initial segment from u to D_{n+1} is the seeked path.

To check now that the two paths are distinct in u, it is enough to notice that in the construction we keep a non empty part of the ϵ, because u does not belong to D_{n+1}.

This ends the inductive part of the construction, so the theorem is proved.

This proof provides an effective procedure to find a pair contradicting H3, i.e. a splitting pair in S. □

The Theorem3.2.10 follows trivially by the previous one, as *ACC* implies *Weak-ACC*.