Corrigé du TD de logique n°2

Logique propositionnelle intuitionniste – Déduction naturelle

Exercice 1 : ***Un raisonnement non intuitionniste***

Si $\sqrt{2}^{\sqrt{2}}$ est rationnel, $a = b = \sqrt{2}$ conviennent. Sinon $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = 2$, et donc $a = \sqrt{2}^{\sqrt{2}}$ et $b = \sqrt{2}$ conviennent. Mais comment savoir lequel est le bon ?

Exercice 2 : ***Quelques démonstrations en logique minimale de Hilbert***

1°)

$$\dfrac{\dfrac{\dfrac{}{\vdash (\varphi\to(\varphi\to\varphi)\to\varphi)\to(\varphi\to(\varphi\to\varphi))\to\varphi\to\varphi}\,\text{(S)} \quad \dfrac{}{\vdash \varphi\to(\varphi\to\varphi)\to\varphi}\,\text{(K)}}{\vdash (\varphi\to(\varphi\to\varphi))\to\varphi\to\varphi}\,(m.p.) \quad \dfrac{}{\vdash \varphi\to\varphi\to\varphi}\,\text{(K)}}{\vdash \varphi\to\varphi}\,(m.p.)$$

2°)

$$\dfrac{\dfrac{\dfrac{}{\vdash ((\alpha\to\beta)\to\psi)\to\varphi}\,\text{(S)} \quad \dfrac{\dfrac{}{\vdash \psi\to(\alpha\to\beta)\to\psi}\,\text{(K)} \quad \dfrac{}{\vdash \psi}\,\text{(S)}}{\vdash (\alpha\to\beta)\to\psi = \vdash (\alpha\to\beta)\to(\gamma\to\alpha\to\beta)\to(\gamma\to\alpha)\to\gamma\to\beta}}{\vdash \varphi = \vdash ((\alpha\to\beta)\to\gamma\to(\alpha\to\beta))\to(\alpha\to\beta)\to(\gamma\to\alpha)\to\gamma\to\beta}\,(m.p.) \quad \dfrac{}{\vdash (\alpha\to\beta)\to\gamma\to(\alpha\to\beta)}\,\text{(K)}}{\vdash (\alpha\to\beta)\to(\gamma\to\alpha)\to\gamma\to\beta}\,(m.p.)$$

En effet, $((\alpha\to\beta)\to\psi)\to\varphi = ((\alpha\to\beta)\to(\gamma\to\alpha\to\beta)\to(\gamma\to\alpha)\to\gamma\to\beta)\to((\alpha\to\beta)\to\gamma\to(\alpha\to\beta))\to(\alpha\to\beta)\to(\gamma\to\alpha)\to\gamma\to\beta$ et $\psi = (\gamma\to\alpha\to\beta)\to(\gamma\to\alpha)\to\gamma\to\beta$ sont bien des instances de (S).

3°)

$$\dfrac{\dfrac{\dfrac{}{\vdash ((\alpha\to\beta)\to(\alpha\to\gamma))\to(\beta\to\alpha\to\beta)\to\beta\to\alpha\to\gamma}\,\text{(B)} \quad \dfrac{\dfrac{}{\vdash (\alpha\to\beta\to\gamma)\to(\alpha\to\beta)\to\alpha\to\gamma}\,\text{(S)} \quad \dfrac{\textit{prémisse}}{\vdash \alpha\to\beta\to\gamma}}{\vdash (\alpha\to\beta)\to\alpha\to\gamma}\,(m.p.)}{\vdash (\beta\to\alpha\to\beta)\to\beta\to\alpha\to\gamma}\,(m.p.) \quad \dfrac{}{\vdash \beta\to\alpha\to\beta}\,\text{(K)}}{\vdash \beta\to\alpha\to\gamma}\,(m.p.)$$

4°)

$$\dfrac{\dfrac{\dfrac{}{\vdash (\beta\to\gamma)\to(\alpha\to\beta)\to\alpha\to\gamma}\,\text{(B)} \quad \dfrac{\textit{prémisse}}{\vdash \beta\to\gamma}}{\vdash (\alpha\to\beta)\to\alpha\to\gamma}\,(m.p.) \quad \dfrac{\textit{prémisse}}{\vdash \alpha\to\beta}}{\vdash \alpha\to\gamma}\,(m.p.)$$

Exercice 3 : ***Des théorèmes passionnants***

1°)

$$\dfrac{\dfrac{}{\vdash (\alpha\to\beta)\to\alpha\to\beta}\,\text{(I)}}{\vdash \alpha\to(\alpha\to\beta)\to\beta}\,\text{(C)}$$

2°)

$$\dfrac{\dfrac{}{\vdash (\alpha\to\beta\to\alpha)\to(\alpha\to\beta)\to\alpha\to\alpha}\,\text{(S)} \quad \dfrac{}{\vdash (\alpha\to\beta\to\alpha)}\,\text{(K)}}{\vdash (\alpha\to\beta)\to\alpha\to\alpha}\,(m.p.) \qquad \text{ou} \qquad \dfrac{\dfrac{}{\vdash \alpha\to(\alpha\to\beta)\to\alpha}\,\text{(K)}}{\vdash (\alpha\to\beta)\to\alpha\to\alpha}\,\text{(C)}$$

3°)

$$\dfrac{\dfrac{}{\vdash \beta\to(\alpha\to\beta)}\,\text{(K)} \qquad \dfrac{\dfrac{}{\vdash (\alpha\to\beta\to\gamma)\to(\alpha\to\beta)\to\alpha\to\gamma}\,\text{(S)}}{\vdash (\alpha\to\beta)\to(\alpha\to\beta\to\gamma)\to\alpha\to\gamma}\,\text{(C)}}{\dfrac{\vdash \beta\to(\alpha\to\beta\to\gamma)\to\alpha\to\gamma}{\vdash (\alpha\to\beta\to\gamma)\to\beta\to\alpha\to\gamma}\,\text{(C)}}\,\text{(cut)}$$

4°)

$$\dfrac{\dfrac{}{\vdash (\beta\to\gamma)\to(\alpha\to\beta)\to\alpha\to\gamma}\,\text{(B)}}{\vdash (\alpha\to\beta)\to(\beta\to\gamma)\to\alpha\to\gamma}\,\text{(C)}$$

5°) On introduit une nouvelle règle : $\dfrac{\vdash \chi\to\psi}{\vdash (\varphi\to\chi)\to\varphi\to\psi}\,\maltese$

Cette règle est correcte dans LM : $\dfrac{\dfrac{}{\vdash (\chi\to\psi)\to(\varphi\to\chi)\to\varphi\to\psi}\,\text{(B)} \qquad \dfrac{\textit{prémisse}}{\vdash \chi\to\psi}}{\vdash (\varphi\to\chi)\to\varphi\to\psi}\,(m.p.)$

On a ensuite, avec $\varphi = (\beta\to\alpha\to\gamma\to\delta)\to(\beta\to\alpha)\to\beta\to\gamma\to\delta$:

$$\dfrac{\dfrac{3°}{\vdash (\alpha\to\beta\to\gamma\to\delta)\to\beta\to\alpha\to\gamma\to\delta} \qquad \dfrac{\dfrac{\dfrac{}{\vdash (\beta\to\gamma\to\delta)\to(\beta\to\gamma)\to\beta\to\delta}\,\text{(S)}}{\vdash ((\beta\to\alpha)\to\beta\to\gamma\to\delta)\to(\beta\to\alpha)\to(\beta\to\gamma)\to\beta\to\delta}\,\maltese \qquad \dfrac{}{\vdash \varphi}\,\text{(S)}}{\vdash (\beta\to\alpha\to\gamma\to\delta)\to(\beta\to\alpha)\to(\beta\to\gamma)\to\beta\to\delta}\,\text{(cut)}}{\vdash (\alpha\to\beta\to\gamma\to\delta)\to(\beta\to\alpha)\to(\beta\to\gamma)\to\beta\to\delta}\,\text{(cut)}$$

Exercice 4 : ***Le connecteur*** $\bot$

1°)

$$\dfrac{\dfrac{\dfrac{}{\vdash (\beta\to\bot)\to(\alpha\to\beta)\to(\alpha\to\bot)}\,\text{(B)}}{\vdash (\alpha\to\beta)\to(\beta\to\bot)\to(\alpha\to\bot)}\,\text{(C)}}{\vdash (\alpha\to\beta)\to(\neg\beta\to\neg\alpha)}\,\text{déf.} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \beta\to\alpha\to\beta}\,\text{(K)} \qquad \dfrac{\textit{prémisse}}{\vdash \beta}}{\vdash \alpha\to\beta}\,(m.p.) \qquad \dfrac{\textit{prémisse}}{\vdash \beta\to\bot}}{\vdash \alpha\to\bot}\,\text{(Cut)}$$

2°)

$$\dfrac{\dfrac{\dfrac{}{\vdash (\beta\to\bot)\to(\alpha\to\bot)\to\alpha\to\beta}\,\text{(B)} \qquad \dfrac{}{\vdash \beta\to\bot}\,\text{(F)}}{\vdash (\alpha\to\bot)\to\alpha\to\beta}\,(m.p.) \qquad \dfrac{}{\vdash \beta\to\alpha\to\beta}\,\text{(K)}}{\vdash (\neg\alpha\vee\beta)\to\alpha\to\beta}\,\text{(}\vee\text{3 – à remplacer)}$$

$$\dfrac{\dfrac{\dfrac{\text{prémisse}}{\beta\to\bot} \qquad \dfrac{}{\vdash \bot\to\alpha}\,\text{(F)}}{\vdash \beta\to\alpha}\,\text{(Cut)} \qquad \dfrac{\text{prémisse}}{\beta}}{\vdash \alpha}\,(m.p.)$$

Exercice 5 : ***Logique propositionnelle intuitionniste***

1°)

Pour montrer (Drop) : $\dfrac{\dfrac{}{\vdash \beta\to\alpha\to\beta}\,\text{(K)} \qquad \dfrac{\textit{prémisse}}{\vdash \beta}}{\vdash \alpha\to\beta}\,(m.p.)$

Pour montrer (Double) : $\dfrac{\dfrac{\dfrac{}{\vdash (\alpha\to\alpha\to\beta)\to(\alpha\to\alpha)\to\alpha\to\beta}\,\text{(S)} \qquad \dfrac{\textit{prémisse}}{\vdash \alpha\to\alpha\to\beta}}{\vdash (\alpha\to\alpha)\to\alpha\to\beta}\,(m.p.) \qquad \dfrac{}{\vdash \alpha\to\alpha}\,\text{(I)}}{\vdash \alpha\to\beta}\,(m.p.)$

Pour montrer $\wedge 3$:

$$\dfrac{\dfrac{\dfrac{\textit{prémisse}}{\vdash \gamma\to\beta} \quad \dfrac{\dfrac{\dfrac{\textit{prémisse}}{\vdash \gamma\to\alpha} \quad \dfrac{}{\vdash \alpha\to\beta\to\alpha\wedge\beta}\,\wedge 0}{\vdash \gamma\to\beta\to\alpha\wedge\beta}\,\text{(Cut)}}{\vdash \beta\to\gamma\to\alpha\wedge\beta}\,\text{(C)}}{\vdash \gamma\to\gamma\to\alpha\wedge\beta}\,\text{(Cut)}}{\vdash \gamma\to\alpha\wedge\beta}\,\text{(Double)}$$

Pour montrer $\vee 3$:

$$\dfrac{\dfrac{\dfrac{}{\vdash (\alpha\to\gamma)\to(\beta\to\gamma)\to(\alpha\vee\beta)\to\gamma}\,\vee 0 \quad \dfrac{\textit{prémisse}}{\vdash \alpha\to\gamma}}{\vdash (\beta\to\gamma)\to(\alpha\vee\beta)\to\gamma}\,(\textit{m.p.}) \quad \dfrac{\textit{prémisse}}{\vdash \beta\to\gamma}}{\vdash (\alpha\vee\beta)\to\gamma}\,(\textit{m.p.})$$

2°)

$$\dfrac{\dfrac{}{\vdash \alpha\to\beta\vee\alpha}\,\vee 2 \quad \dfrac{}{\vdash \beta\to\beta\vee\alpha}\,\vee 1}{\vdash \alpha\vee\beta\to\beta\vee\alpha}\,\vee 3$$

$$\dfrac{\dfrac{\dfrac{}{\vdash (\beta\to\beta\vee\gamma)\to(\alpha\to\beta)\to\alpha\to(\beta\vee\gamma)}\,\text{(B)} \quad \dfrac{}{\beta\to\beta\vee\gamma}\,\wedge 1}{\vdash (\alpha\to\beta)\to\alpha\to(\beta\vee\gamma)}\,(\textit{m.p.}) \quad \dfrac{\dfrac{}{\vdash \ldots}\,\text{C+}\vee 0\,\dfrac{}{\gamma\to(\beta\vee\gamma)}\,\vee 2}{\vdash (\alpha\to(\beta\vee\gamma))\to(\alpha\vee\gamma)\to(\beta\vee\gamma)}\,(\textit{m.p.})}{\vdash (\alpha\to\beta)\to(\alpha\vee\gamma)\to(\beta\vee\gamma)}\,\text{(Cut)}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash \alpha\to\alpha\vee\beta}\,\vee 1\,\dfrac{}{\vdash \alpha\to\alpha\vee\gamma}\,\vee 1}{\vdash \alpha\to(\alpha\vee\beta)\wedge(\alpha\vee\gamma)}\,\wedge 3 \quad \dfrac{\dfrac{\dfrac{}{\vdash \beta\wedge\gamma\to\beta}\,\wedge 1 \quad \dfrac{}{\vdash \beta\to\alpha\vee\beta}\,\vee 2}{\vdash \beta\wedge\gamma\to\alpha\vee\beta}\,\text{(Cut)} \quad \dfrac{\dfrac{}{\vdash \beta\wedge\gamma\to\gamma}\,\wedge 2\,\dfrac{}{\vdash \gamma\to\alpha\vee\gamma}\,\vee 2}{\vdash \beta\wedge\gamma\to\alpha\vee\gamma}\,\text{(Cut)}}{\vdash \beta\wedge\gamma\to(\alpha\vee\beta)\wedge(\alpha\vee\gamma)}\,\wedge 3}{\vdash \alpha\vee(\beta\wedge\gamma)\to(\alpha\vee\beta)\wedge(\alpha\vee\gamma)}\,\vee 3$$

3°)

$$\dfrac{\dfrac{\dfrac{}{\vdash \alpha\wedge\beta\to\alpha}\,\wedge 1 \quad \dfrac{\textit{prémisse}}{\vdash \alpha\to\gamma}}{\vdash \alpha\wedge\beta\to\gamma}\,\text{Cut} \quad \dfrac{\dfrac{}{\vdash \alpha\wedge\beta\to\beta}\,\wedge 2 \quad \dfrac{\textit{prémisse}}{\vdash \beta\to\delta}}{\vdash \alpha\wedge\beta\to\delta}\,\text{Cut}}{\vdash \alpha\wedge\beta\to\gamma\wedge\delta}\,\wedge 3$$