Second Order Types

Loss of information

Consider the function

$$I = \lambda x^T.x \ : \ T \to T$$

By the rule for application

$$\frac{I : T \to T \qquad M : U < T}{I(M) : T}$$

Therefore

$$(\lambda x^{\langle\!\langle a:int \rangle\!\rangle}.x)\langle a = 1, b = 2\rangle : \langle\!\langle a : int \rangle\!\rangle$$

Second order

$$I \ : \ \forall X \leq T.X \to X$$

Two ways:

1. Implicit polymorphism
2. Explicit polymorphism

Implicit polymorphism

No types in terms

$\lambda x.x : \forall\alpha.\alpha \to \alpha$ $\qquad\qquad$ $(\lambda x.x)3 : int$

$$\dfrac{[\alpha = \beta]\;\dfrac{x:\alpha \vdash x:\alpha}{\vdash \lambda x.x : \alpha \to \beta} \qquad \vdash 3 : int}{\vdash (\lambda x.x)3 : \beta}\;[\alpha = int]$$

Subtyping

$$\lambda x.((\lambda y.x)(x.\ell + 3)) : \forall\alpha \leq \langle\!\langle \ell : int \rangle\!\rangle.\alpha \to \alpha$$

Therefore

$$\lambda x.((\lambda y.x)(x.\ell{+}3))(\langle \ell = 1, m = true \rangle) : \langle\!\langle \ell : int, m : bool \rangle\!\rangle$$

Inference with subtyping

$$
\dfrac{\dfrac{x{:}\alpha, y{:}\gamma \vdash x : \alpha}{x{:}\alpha \vdash \lambda y.x : \gamma \to \beta}\ \alpha = \beta \qquad \dfrac{x{:}\alpha \vdash 3 : int \qquad \dfrac{x{:}\alpha \vdash x{:}\alpha}{x{:}\alpha \vdash x.\ell : \epsilon}\ \alpha \leq \langle\!\langle \ell : \epsilon \rangle\!\rangle}{x{:}\alpha \vdash x.\ell + 3 : \delta}\ \delta = int, \epsilon \leq int}{\dfrac{x{:}\alpha \vdash (\lambda y.x)(x.\ell + 3) : \beta}{\vdash \lambda x.((\lambda y.x)(x.\ell + 3)) : \alpha \to \beta}}\ \delta \leq \gamma
$$

Resulting type

$$\forall \epsilon \leq int \,.\, \forall \alpha \leq \langle\!\langle \ell{:}\epsilon \rangle\!\rangle \,.\, \alpha \to \alpha$$

Simplified

$$\forall \alpha \leq \langle\!\langle \ell{:}int \rangle\!\rangle \,.\, \alpha \to \alpha$$

Explicit polymorphism

$$\Lambda X.\lambda x^X.x \ : \ \forall X.X \to X$$

The programmer specifies the type

$$(\Lambda X.\lambda x^X.x)(int)(3) \quad \triangleright \quad (\lambda x^{int}.x)(3)$$

Subtyping

$$\Lambda X \leq \langle\!\langle a{:}int \rangle\!\rangle.\lambda x^X.x$$

The application

$$(\Lambda X \leq \langle\!\langle a{:}int \rangle\!\rangle.\lambda x^X.x)(\langle\!\langle a{:}int, b{:}int \rangle\!\rangle)$$

has type

$$\langle\!\langle a{:}int, b{:}int \rangle\!\rangle \to \langle\!\langle a{:}int, b{:}int \rangle\!\rangle$$

thus

$$(\Lambda X \leq \langle\!\langle a{:}int \rangle\!\rangle.\lambda x^X.x)(\langle\!\langle a{:}int, b{:}int \rangle\!\rangle)(\langle a = 1, b = 3 \rangle)$$

has type

$$\langle\!\langle a{:}int, b{:}int \rangle\!\rangle$$

$F_{\leq}$

Types

$$T \; ::= \; X \mid \textbf{Top} \mid T \to T \mid \forall(X{\leq}T)T$$

Terms

$$\begin{array}{rcl} a & ::= & x \mid (\lambda x^T.a) \mid a(a) \\ & \mid & \textbf{top} \mid \Lambda X{\leq}T.a \mid a(T) \end{array}$$

Reduction

$$(\beta) \qquad (\lambda x^T.a)(b) \triangleright a[x^T := b]$$

$$(\beta_\forall) \qquad (\Lambda X{\leq}T.a)(T') \triangleright a[X := T']$$

Subtyping

(refl) $$C \vdash T \leq T$$

(trans) $$\frac{C \vdash T_1 \leq T_2 \quad C \vdash T_2 \leq T_3}{C \vdash T_1 \leq T_3}$$

(taut) $$C \vdash X \leq C(X)$$

(Top) $$C \vdash T \leq \mathbf{Top}$$

($\rightarrow$) $$\frac{C \vdash T_1 \leq S_1 \quad C \vdash S_2 \leq T_2}{C \vdash S_1 \rightarrow S_2 \leq T_1 \rightarrow T_2}$$

($\forall$) $$\frac{C \vdash T_1 \leq S_1 \quad C, (X \leq T_1) \vdash S_2 \leq T_2}{C \vdash \forall (X \leq S_1) S_2 \leq \forall (X \leq T_1) T_2}$$

Type system

[Vars] $$C\,;\,\Gamma \vdash x\colon \Gamma(x)$$

[→Intro] $$\frac{C\,;\,\Gamma,(x\colon T) \vdash a\colon T'}{C\,;\,\Gamma \vdash (\lambda x^T.a)\colon T \to T'}$$

[→Elim] $$\frac{C\,;\,\Gamma \vdash a\colon S \to T \qquad C\,;\,\Gamma \vdash b\colon S}{C\,;\,\Gamma \vdash a(b)\colon T}$$

[Top] $$C\,;\,\Gamma \vdash \mathbf{top}\colon \mathbf{Top}$$

[∀Intro] $$\frac{C,(X{\leq}T)\,;\,\Gamma \vdash a\colon T'}{C\,;\,\Gamma \vdash \Lambda X{\leq}T.a\colon \forall(X{\leq}T)T'}$$

[∀Elim] $$\frac{C\,;\,\Gamma \vdash a\colon \forall(X{\leq}S)T}{C\,;\,\Gamma \vdash a(S)\colon T[X := S]}$$

[Subsump] $$\frac{C\,;\,\Gamma \vdash a\colon T' \qquad C \vdash T' \leq T}{C\,;\,\Gamma \vdash a\colon T}$$

Transitivity elimination

$$c \;::=\; Id_A \mid X_T \mid \mathbf{Top}_T \mid c \to c' \mid \forall(X{\leq}c)c' \mid c\,c'$$

(refl) $$C \vdash Id_A\colon A \leq A$$

(trans) $$\frac{C \vdash c\colon T_1 \leq T_2 \qquad C \vdash c'\colon T_2 \leq T_3}{C \vdash c'\,c\colon T_1 \leq T_3}$$

(taut) $$C \cup \{X{\leq}T\} \vdash X_T\colon X \leq T$$

(Top) $$C \vdash \mathbf{Top}_T\colon T \leq \mathbf{Top}$$

($\to$) $$\frac{C \vdash c_1\colon T_1' \leq T_1 \qquad C \vdash c_2\colon T_2 \leq T_2'}{C \vdash c_1 \to c_2\colon T_1 \to T_2 \leq T_1' \to T_2'}$$

($\forall$) $$\frac{C \vdash c_1\colon T_1' \leq T_1 \qquad C \cup \{X{\leq}T_1'\} \vdash c_2\colon T_2 \leq T_2'}{C \vdash \forall(X{\leq}c_1)c_2\colon \forall(X{\leq}T_1)T_2 \leq \forall(X{\leq}T_1')T_2'}$$

Theorem 5 *There is a 1-1 correspondence between well-typed coerce expressions and subtyping derivations.*

The rewriting system

(Asc)	$(c\,d)\,e$	$\rightsquigarrow$	$c\,(d\,e)$
$(\to')$	$(c \to d)\,(c' \to d')$	$\rightsquigarrow$	$(c'\,c) \to (d\,d')$
$(\to'')$	$(c \to d)\,((c' \to d')\,e)$	$\rightsquigarrow$	$((c'\,c) \to (d\,d'))\,e$
$(\forall')$	$(\forall(X{\leq}c)d)\,(\forall(X{\leq}c')d')$	$\rightsquigarrow$	$\forall(X{\leq}c'\,c)(d\,d'[X_T{:=}\,c\,X_S])$
$(\forall'')$	$(\forall(X{\leq}c)d)\,((\forall(X{\leq}c')d')\,e)$	$\rightsquigarrow$	$(\forall(X{\leq}c'\,c)(d\,d'[X_T{:=}\,c\,X_S]))e$

Normal forms are subterms of $(c \to d)\,e_1\,\ldots\,e_n$ or of $(\forall(X{\leq}c)d)\,e_1\,\ldots\,e_n$ where c, c_i, d, d_i are in normal form and $e_1, \ldots, e_n$ are either X_t or $\mathbf{Top}_T$. They normal forms correspond to derivations in which every left premise of a (trans) rule is a leaf. Thus, the rewriting system pushes the transitivity up to the leaves.

Example

$$(c \to d)\,((c' \to d')\,e) \rightsquigarrow ((c'\,c) \to (d\,d'))\,e$$

Theorem 6 (Soundness) *If* $c \overset{*}{\rightsquigarrow} d$ *and* $C \vdash c\colon \Delta$ *then* $C \vdash d\colon \Delta$

Theorem 7 (Weak normalization) *Every innermost strategy for* $\rightsquigarrow$ *terminates.*

Coherence

Let $c : S \leq T$

(id_l)	$Id_T\, c$	$\leadsto$	c
(id_r)	$c\, Id_S$	$\leadsto$	c
(top)	$\mathbf{Top}_T\, c$	$\leadsto$	$\mathbf{Top}_S$
(varTop)	X_{Top}	$\leadsto$	$\mathbf{Top}_X$

Consider the composition of the rewriting systems:

Theorem 8 (normal forms) *Every well-typed coerce expression in normal form has the form $c_0\, c_1 \ldots c_n$ with $n \geq 0$, where c_0 can be any coerce expression different from a composition (of other coerce expressions) whose subformulae are in normal form, and $c_1 \ldots c_n$ are variables.*

Theorem 9 *For every provable subtyping judgment, there exists only one coerce expression in normal form proving it.*

Coherence

Theorem 10 (coherence) *Let Π_1 and Π_2 be two proofs of the same judgment $C \vdash \Delta$. If c_1 and c_2 are the corresponding coerce expressions then c_1 and c_2 are equal modulo the rewriting system.*

Shape of NFs and the subtyping algorithm

The normal forms of Theorem 8 correspond to derivations in which every application of a (trans) rule has as left premise an application of the rule (taut).

Subtyping algorithm

(AlgRefl) $$C \vdash X \leq X$$

(AlgTrans) $$\frac{C \vdash C(X) \leq T}{C \vdash X \leq T}$$

(Top) $$C \vdash T \leq \mathbf{Top}$$

($\rightarrow$) $$\frac{C \vdash T_1 \leq S_1 \qquad C \vdash S_2 \leq T_2}{C \vdash S_1 \rightarrow S_2 \leq T_1 \rightarrow T_2}$$

($\forall$) $$\frac{C \vdash T_1 \leq S_1 \qquad C, (X \leq T_1) \vdash S_2 \leq T_2}{C \vdash \forall (X \leq S_1) S_2 \leq \forall (X \leq T_1) T_2}$$

Typing algorithm

[Vars] $$C\,;\,\Gamma \vdash x\colon \Gamma(x)$$

[$\rightarrow$I] $$\frac{C\,;\,\Gamma,(x\colon T) \vdash a\colon T'}{C\,;\,\Gamma \vdash (\lambda x^T.a)\colon T \rightarrow T'}$$

[$\rightarrow$E] $$\frac{C;\Gamma \vdash a\colon U \quad C;\Gamma \vdash b\colon S' \quad C \vdash S' \leq S}{C;\Gamma \vdash a(b)\colon T} \quad \mathscr{B}_C(U) = S \rightarrow T$$

[Top] $$C\,;\,\Gamma \vdash \mathbf{top}\colon \mathbf{Top}$$

[$\forall$I] $$\frac{C,(X \leq T)\,;\,\Gamma \vdash a\colon T'}{C\,;\,\Gamma \vdash \Lambda X \leq T.a\colon \forall(X \leq T)T'}$$

[$\forall$E] $$\frac{C\,;\,\Gamma \vdash a\colon U \quad C \vdash S' \leq S}{C\,;\,\Gamma \vdash a(S')\colon T[X := S']} \quad \mathscr{B}_C(U) = \forall(X \leq S)T$$

Definition 2

$$\mathscr{B}_C(T) = \begin{cases} \mathscr{B}_C(C(X)) & \textit{if } T \equiv X \\ T & \textit{otherwise} \end{cases}$$

Typing and subtyping algorithms are sound and complete

Sound and complete does not mean decidable!!

let $\neg T$ and $\forall(X)T$ denote $T \to \mathbf{Top}$ and $\forall(X{\leq}\mathbf{Top})T$:

$$X_0 \leq \forall(Y)\neg(\forall(Z \leq Y)\neg Y) \vdash X_0 \leq \forall(X_1{\leq}X_0)\neg X_0$$

by applying AlgTrans:

$$X_0 \leq \forall(Y)\neg(\forall(Z \leq Y)\neg Y) \vdash \forall(X_1)\neg(\forall(X_2{\leq}X_1)\neg X_1) \leq \forall(X_1{\leq}X_0)\neg X_0$$

by applying $(\forall)$:

$$X_0 \leq \forall(Y)\neg(\forall(Z \leq Y)\neg Y), X_1{\leq}X_0 \vdash \neg(\forall(X_2{\leq}X_1)\neg X_1) \leq \neg X_0$$

by the contravariance of $(\to)$:

$$X_0 \leq \forall(Y)\neg(\forall(Z \leq Y)\neg Y), X_1{\leq}X_0 \vdash X_0 \leq \forall(X_2{\leq}X_1)\neg X_1$$

the same judgement as the one we started from.

Just semi-decidability holds

Kernel-Fun: compare quantifications with equal bounds.