Strategies

Strategy

Définition : A reduction strategy (one-steps or multi-step) for a rewriting system $\mathcal{R}$ is a function $I\!\!F : \mathcal{T}(\mathcal{X}, \Sigma) \mapsto \mathcal{T}(\mathcal{X}, \Sigma)$ s.t.

1. $I\!\!F(t) = t$ if t is in $\mathcal{R}$-normal form.
2. $t \rightarrow^{+} I\!\!F(t)$ otherwise.

weakly normalizing

$I\!\!F$ is normalizing iff for every WN term t there is no infinite sequence $t \rightarrow I\!\!F(t) \rightarrow I\!\!F(I\!\!F(t)) \rightarrow I\!\!F(I\!\!F(I\!\!F(t))) \rightarrow \ldots$

In what follows, we will focus only on orthogonal systems.

Classification

- Strategies without history
  - Innermost
  - Leftmost-innermost
  - Parallel-innermost
  - Leftmost-outermost (standard)
  - Parallel-outermost
  - Complete
- Strategies with history
  - Complete development
  - Standard

Strategies without history

Innermost Strategy(s)

The innermost strategy rewrites ONE redex of the set of all the innermost redexes.

Example :

$$\mathcal{R}\left\{\begin{array}{lllllll} f(a,x) & \rightarrow & x & \quad & f(b,x) & \rightarrow & b \\ g(a,x) & \rightarrow & a & \quad & g(b,x) & \rightarrow & x \end{array}\right.$$

$$f(f(a,f(a,b)),g(f(a,b),g(b,a)))$$

$$f(f(a,f(a,b)),g(f(a,b),g(b,a)))$$

$$f(f(a,f(a,b)),g(f(a,b),g(b,a)))$$

Leftmost-innermost strategy

The leftmost-innermost strategy rewrites the leftmost redex of the set of the innermost redexes.

Example :

$$\mathcal{R}\left\{\begin{array}{lcl} f(a,x) & \rightarrow & x \\ f(b,x) & \rightarrow & b \\ g(a,x) & \rightarrow & a \\ g(b,x) & \rightarrow & x \end{array}\right.$$

$$f(f(a, f(a,b)), g(f(a,b), g(b,a)))$$

Parallel-innermost strategy

The parallel-innermost strategy rewrites simultaneously ALL the innermost redexes.

$$f(f(a, f(a, b)), g(f(a, b), g(b, a)))$$

Remarque : There is only one "leftmost-innermost" or "parallel-innermost" strategy but many "innermost" strategies.

Leftmost-outermost strategy

The leftmost-outermost strategy rewrites the leftmost redex of all the set of outermost redexes.

$$f(f(a, f(a, b)), g(f(a, b), g(b, a)))$$

Parallel-outermost strategy

The parallel-outermost strategy rewrites simultaneously the set of ALL the outermost redexes.

$$f(f(a, f(a, b)), g(f(a, b), g(b, a)))$$

Complete strategy

The complete strategy rewrites simultaneously all the redexes.

$$f(\underline{f(a, \underline{f(a,b)})}, g(\underline{f(a,b)}, \underline{g(b,a)}))$$

Are these strategies normalizing ?

The leftmost-innermost strategy is not normalizing

Example : Let $\mathcal{R}$ be the following system :

$$\mathcal{R} \left\{ \begin{array}{lcl} f(x,b) & \rightarrow & d \\ a & \rightarrow & b \\ c & \rightarrow & c \end{array} \right.$$

$$f(\underline{c}, a) \rightarrow f(\underline{c}, a) \rightarrow \ldots$$

The parallel-innermost strategy is not normalizing

Example : Let $\mathcal{R}$ be the following system :

$$\mathcal{R} \begin{cases} f(x,b) & \longrightarrow & d \\ a & \longrightarrow & b \\ c & \longrightarrow & c \end{cases}$$

$$f(\underline{c}, \underline{a}) \rightarrow f(\underline{c}, b) \rightarrow \ldots$$

In general...

Theorem : Let $\mathcal{R}$ an orthogonal system. Then the innermost strategy is weakly normalizing for $\mathcal{R}$ iff $\mathcal{R}$ is strongly normalizing.

The leftmost-outermost strategy is not normalizing

Example :

$$\mathcal{R} \left\{ \begin{array}{lcl} f(x, b) & \rightarrow & d \\ a & \rightarrow & b \\ c & \rightarrow & c \end{array} \right.$$

$$f(\underline{c}, a) \rightarrow f(\underline{c}, a) \rightarrow \ldots$$

Making the leftmost-outermost strategy normalizing

Most of the interesting (functional) programs are left normal.

Définition : A term is left norm [illegible] all the function symbols appear before the variables. A system $\mathcal{R}$ is left normal iff for every rule $l \to r \in \mathcal{R}$ the term l is left normal.

Example :

The term $f(x, b)$ is not left normal, the term $f(b, x)$ is left normal.

Theorem : The leftmost-outermost strategy is normalizing for all the orthogonal systems which are left normal.

Parallel Outermost

The parallel-outermost strategy is normalizing for orthogonal systems.

Complete Strategy

The complete strategy is normalizing for orthogonal systems.

Strategies with history

Residual theory

J-J. Lévy

Descendants

$$\left\{ \begin{array}{l} f(x) \to h(x,x) \\ g(a) \to a \end{array} \right.$$

$$\begin{array}{ccc} f(g(a)) & \longrightarrow & h(g(a), g(a)) \\ & & \downarrow \\ & & h(a, g(a)) \\ \downarrow & & \downarrow \\ f(a) & \longrightarrow & h(a, a) \end{array}$$

Accidents

$$\mathcal{R} : I(x) \to x$$

Where does the reduction $\mathcal{R}$ take place in $I(I(x)) \to I(x)$?

$$\mathcal{R} : a \to a$$

Where does the reduction $\mathcal{R}$ take place in $f(a, a) \to f(a, a)$?

Label terms

We need to specify the rule which is used, and the position in which reduction takes place.

$$f(g(a)) \to_{f(g(a))} f(a)$$

$$f(g(a)) \to_{f(g(a))} h(g(a), g(a)) \to_{h(g(a),g(a))} h(a, g(a))$$

$$I(I(x)) \to_{I(I(x))} I(x) \qquad I(I(x)) \to_{I(I(x))} I(x)$$

$$f(a, a) \to_{f(a,a)} f(a, a) \qquad f(a, a) \to_{f(a,a)} f(a, a)$$

Redex trace

Let s and s' two redexes of u s.t. $u \equiv t[s] \rightarrow_{v[s']} u'$.

What happens with s in u' after the reduction of s' ?

- If s and s' are disjoints : s appears inside u'.
- If s and s' are the same : the redex s is erased in u'.
- If s' is a strict subterm of s : since s' is an argument of s, so that s appears in u' with a different argument.
- If s is a strict subterm of s' : s appears $n \geq 0$ times in u'.
- All the other redexes in u' are created.

Example

$$\mathcal{R}\left\{\begin{array}{lll} a & \rightarrow_1 & b \\ b & \rightarrow_2 & c \\ c & \rightarrow_3 & d \\ f(x,y) & \rightarrow_4 & g(y,y) \\ g(x,h(y)) & \rightarrow_5 & h(y) \end{array}\right.$$

$$t : \underbrace{g(\underbrace{f(\underbrace{\ c\ }, h(\underbrace{\ a\ }))}, h(\underbrace{\ b\ }))} \rightarrow_4 \underbrace{g(\underbrace{g(h(\underbrace{\ a\ }), h(\underbrace{\ a\ }))}, h(\underbrace{\ b\ }))} : t'$$

- The redex a in t is duplicated twice in t'.
- The redex b in t appears once in t'.
- The redex c in t is erased in t'.
- The redex $f(c, h(a))$ in t has no e descendant in t'.
- The redex $g(f(c, h(a)), h(b))$ in t has a descendant $g(g(h(a), h(a)), h(b))$ in t'.
- The redex $g(h(a), h(a))$ in t' is created.

Descendants and residuals

Remarque :

Let $\mathcal{R}$ an orthogonal system. Descendants of a redex are always redexes.

Définition : A redex which is descendant of a redex is a residual, otherwise it is a created redex.

Residual of a redex

the position of the redex s

Définition : Let s be any redex of t. The set of … w.r.t. the reduction step $\rho : t \rightarrow_{s'} t'$, written $Res(s, \rho)$, is given by :

- If s and s' are disjoints, then $Res(s, \rho) = \{s\}$
- If s and s' are the same, then $Res(s, \rho) = \emptyset$.
- If s is a strict subterm of s', then $Res(s, \rho)$ contains n $(n \geq 0)$ times s.
- If s strictly contains s', then $Res(s, \rho)$ contains s where each occurrence of s' has been replaced by its contractum.

Residual of a set of redexes

Let S a set of redexes of t. The set of residuals of S w.r.t. the reduction $\rho : t \rightarrow_{s'} t'$ is given by

$$Res(S, \rho) = \bigcup_{s \in S} Res(s, \rho)$$

Residual by a reduction sequence

Définition : Let s be any redex in t. The set of residuals of s w.r.t. the reduction sequence $\rho : t \rightarrow^* t'$, written $Res^*(s, \rho)$, is given by :

$$\begin{array}{lcl} Res^*(s, \epsilon) & = & \{s\} \\ Res^*(s, \rho) & = & Res(s, \rho), \ldots \\ Res^*(s, \delta\rho) & = & Res^*(Res(s, \delta), \rho) \end{array}$$

this is a set

where $Res^*(S, \rho)$ is given by :

$$Res^*(S, \rho) = \bigcup_{s \in S} Res^*(s, \rho)$$

Developments

A development of a term t is a reduction sequence where no created redex can be reduced, that is, only residual of the set of redex of t can be reduced. Formally,

Définition :

Let S be the set of all the redexes of t. A development of a term t is a reduction sequence :

$$\rho : t \longrightarrow_{s_0} t_0 \longrightarrow_{s_1} t_1 \longrightarrow_{s_2} t_2 \ldots$$

such that for every s_i we have $s_i \in Res^*(S, s_0 \ldots s_{i-1})$.

Complete Developments

Définition : Let S be the set of all the redexes of t. A development $\rho : t \longrightarrow^* t'$ is complete iff $Res^*(S, \rho) = \emptyset$.

Theorem : The complete development strategy is normalizing.