Abstract Rewriting

Formal definition of rewriting

Given a set of objects A, an (abstract) rewriting system is a relation $\mathcal{R} \subseteq A \times A$.

Example :

A = the set of finite sequences over $\{\circ, \bullet, \bullet\}$.

$$\text{Rewriting system } \mathcal{R} = \begin{cases} \circ\ \bullet & \longrightarrow_1 \quad \bullet\ \circ \\ \bullet\ \bullet & \longrightarrow_2 \quad \bullet\ \bullet \\ \bullet\ \circ & \longrightarrow_3 \quad \circ\ \bullet \end{cases}$$

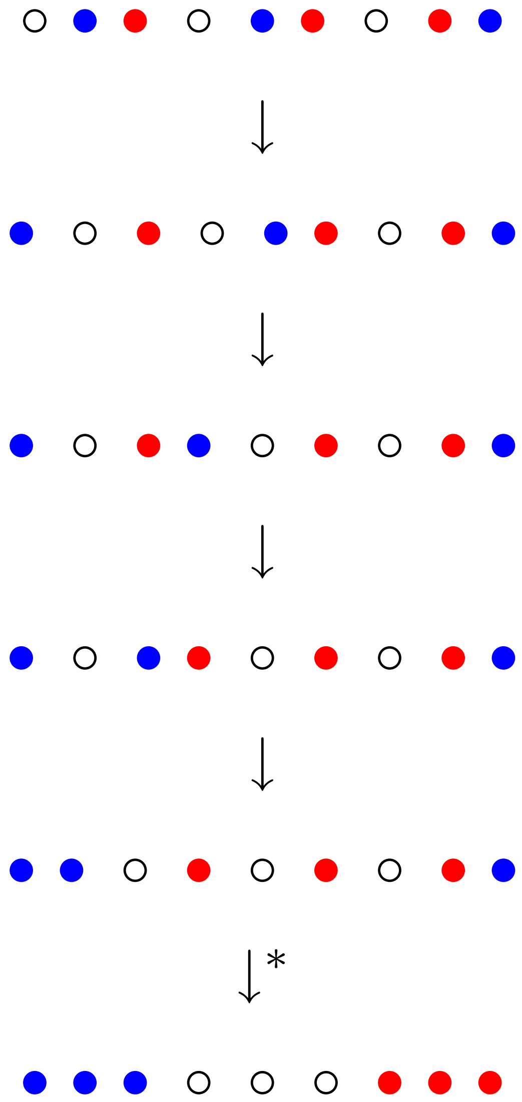
Closure notions

- An $\mathcal{R}$-rewrite sequence has the form

  $s = s_0 \rightarrow_{\mathcal{R}} s_1 \rightarrow_{\mathcal{R}} \ldots \rightarrow_{\mathcal{R}} s_n = t$ (for $n \geq 0$).
- $s \rightarrow^{+}_{\mathcal{R}} t$ is the transitive closure of $\rightarrow_{\mathcal{R}}$.
- $\rightarrow^{=}_{\mathcal{R}}$ is the reflexive closure.
- $\rightarrow^{*}_{\mathcal{R}}$ is the reflexive transitive closure of $\rightarrow_{\mathcal{R}}$.

- $\leftrightarrow_{\mathcal{R}}$ is the symmetrique closure.
- $\leftrightarrow^{*}_{\mathcal{R}}$ is the reflexive, symetrique and transitive closure.

Example of $\mathcal{R}$-rewrite sequence



More basic vocabulary

- A term t is $\mathcal{R}$-reducible iff there exists s s.t. $t \rightarrow_{\mathcal{R}} s$.
- A term t is in $\mathcal{R}$-normal form iff it is no $\mathcal{R}$-reducible.
- A term s is a $\mathcal{R}$-normal form of t iff $t \rightarrow^{*}_{\mathcal{R}} s$ and s is in $\mathcal{R}$-normal form.

Different meaning for equivalent terms

Given again

$$\mathcal{R} = \begin{cases} f(x,x) & \rightarrow & c \\ a & \rightarrow & b \\ f(x,b) & \rightarrow & d \end{cases}$$

we can compute from the same term $f(a,a)$ two different results c and d.

Same meaning for equivalent terms

$\mathcal{R}$ is Church-Rosser iff

$$\begin{array}{ccc} u & \longleftrightarrow^*_{\mathcal{R}} & v \\ & \searrow_{*\mathcal{R}} \quad {}_{*\mathcal{R}}\swarrow & \\ & s & \end{array}$$

Confluence diagrams

A diagram like :

$$
\begin{array}{ccc}
t & \mathcal{R}_1 & u \\
\mathcal{R}_2 & & \mathcal{R}_3 \\
v & \mathcal{R}_4 & s
\end{array}
$$

reads :

for all t, u, v such that $t\ \mathcal{R}_1\ u$ and $t\ \mathcal{R}_2\ v$,
there exist s such that $u\ \mathcal{R}_3\ s$ and $v\ \mathcal{R}_4\ s$.

Confluence notions

– $\mathcal{R}$ is confluent iff

$$\begin{array}{ccc} t & \longrightarrow^*_{\mathcal{R}} & u \\ \downarrow_{*\mathcal{R}} & & \downarrow_{*\mathcal{R}} \\ v & \longrightarrow^*_{\mathcal{R}} & s \end{array}$$

– $\mathcal{R}$ is locally confluent iff

$$\begin{array}{ccc} t & \longrightarrow_{\mathcal{R}} & u \\ \downarrow_{\mathcal{R}} & & \downarrow_{*\mathcal{R}} \\ v & \longrightarrow^*_{\mathcal{R}} & s \end{array}$$

– $\mathcal{R}$ is strongly confluent iff

$$\begin{array}{ccc} t & \rightarrow_{\mathcal{R}} & u \\ \downarrow_{\mathcal{R}} & & \downarrow_{*\mathcal{R}} \\ v & \rightarrow_{\mathcal{R}}^{=} & s \end{array}$$

– $\mathcal{R}$ has the diamond property iff

$$\begin{array}{ccc} t & \rightarrow_{\mathcal{R}} & u \\ \downarrow_{\mathcal{R}} & & \downarrow_{\mathcal{R}} \\ v & \rightarrow_{\mathcal{R}} & s \end{array}$$

This is a particular case of strongly confluence.

Equivalent notions

Theorem : $\mathcal{R}$ is Church-Rosser iff $\mathcal{R}$ is confluent.

Not equivalent notions

The following system [Curry] :

$$\mathcal{R} = \begin{cases} c & \longrightarrow & a \\ c & \longrightarrow & d \\ d & \longrightarrow & c \\ d & \longrightarrow & b \end{cases}$$

is locally confluent but not confluent :

$$a \leftarrow c \longrightarrow^* b$$

Termination notions

- The system $\mathcal{R}$ is weakly normalising (WN) iff every element has at least one $\mathcal{R}$-normal form.
- The system $\mathcal{R}$ terminates or is strongly normalising (SN) or noetherien or well-founded (WF) iff every $\mathcal{R}$-reduction sequence starting at s is finite. We note $s \in SN_{\mathcal{S}}$.

Weak vs strong normalisation

$$\mathcal{R} = \begin{cases} f(a) & \rightarrow & c \\ f(x) & \rightarrow & f(a) \end{cases}$$

The system is weakly normalising but not strongly normalising :

$$f(b) \rightarrow f(a) \rightarrow c$$

$$f(b) \rightarrow f(a) \rightarrow f(a) \ldots$$

Convergent Systems

Définition : The system $\mathcal{R}$ is convergent iff it is confluent and strongly normalising.

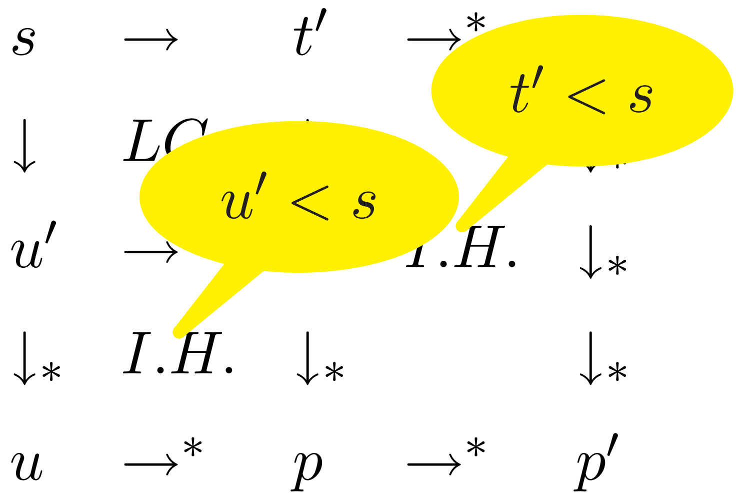
Remarque :

- If $\mathcal{R}$ is confluent, then every element has at most a normal form.
- If $\mathcal{R}$ is convergent, then every element has one and only one normal form. In this case, we use the *functional* notation $\mathcal{R}(t)$ to denote the $\mathcal{R}$-normal form of t.

Confluence from local confluence

Lemme : (Newmann) Let $\mathcal{R}$ be a SN system. Then $\mathcal{R}$ is locally confluent iff $\mathcal{R}$ is confluent.

Proof. (By Huet) By well-founded induction on $s \in SN$.



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Important remark

The following (infinite) system on natural numbers :

$$\mathcal{R} = \begin{cases} 2.n & \rightarrow & 2.n+1 \\ 2.n & \rightarrow & a \\ 2.m+1 & \rightarrow & 2.m+2 \\ 2.m+1 & \rightarrow & b \end{cases}$$

is locally confluent but not confluent : $a \leftarrow 0 \rightarrow^* b$

In fact it is not SN

$$0 \rightarrow\ 1 \rightarrow\ 2 \rightarrow\ 3 \rightarrow\ \ldots$$