
Confluence

Motivations

Confluence is an undecidable property.

Decision procedures

- Finite
- Sound
- Terminating

Some techniques to show confluence

- Confluence by strong confluence
- Confluence by equivalence
- Confluence by commutation
- Confluence by interpretation
- Confluence by critical pairs
- Confluence by orthogonality
- Confluence by decreasing diagrams

Confluence by strong confluence

Theorem : If \mathcal{R} is **strongly confluent**, then \mathcal{R} is **confluent**.

Example :

$$\mathcal{R} = \left\{ \begin{array}{l} f(x, x) \rightarrow g(x) \\ f(x, y) \rightarrow g(y) \\ g(x) \rightarrow f(x, a) \end{array} \right.$$

We only check diagrams for one-step reduction sequences.

Corollaire : If \mathcal{R} has the **diamond property**, then \mathcal{R} is **confluent**.

Confluence by equivalence

Theorem : Let \mathcal{R} and \mathcal{S} two rewrite systems such that $\rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{S}} \subseteq \rightarrow_{\mathcal{R}}^*$ and \mathcal{S} is strongly confluent. Then \mathcal{R} is confluent.

Proof.

- If $\rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{S}} \subseteq \rightarrow_{\mathcal{R}}^*$, then $\rightarrow_{\mathcal{R}}^* = \rightarrow_{\mathcal{S}}^*$.
- If \mathcal{S} is strongly confluent, then \mathcal{S} is confluent.
- If $\rightarrow_{\mathcal{R}}^* = \rightarrow_{\mathcal{S}}^*$, then \mathcal{R} is confluent iff \mathcal{S} is confluent.

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(Famous) Example : confluence of β

Show that $(\lambda x.M)N \rightarrow_{\beta} M\{x/N\}$ is confluent.

Define a relation \gg as follows :

$$\frac{}{x \gg x} \qquad \frac{t \gg t'}{\lambda x.t \gg \lambda x.t'}$$
$$\frac{t \gg t' \text{ and } u \gg u'}{t u \gg t' u'} \qquad \frac{t \gg t' \text{ and } u \gg u'}{(\lambda x.t) u \gg t'\{x/u'\}}$$

Let $\mathcal{R} = \beta$ and let $\mathcal{S} = \gg$. Now,

1. Show that $\beta \subseteq \gg \subseteq \beta^*$.
2. Show that \gg is strongly confluent.
3. Conclude that β is confluent (on all the terms).

Confluence by commutation [Hindley-Rosen]

Two systems \mathcal{R} and \mathcal{S} commute iff

$$\begin{array}{ccc} & \rightarrow^*_{\mathcal{R}} & \\ \downarrow^*_{\mathcal{S}} & & \downarrow^*_{\mathcal{S}} \\ & \rightarrow^*_{\mathcal{R}} & \end{array}$$

Two systems \mathcal{R} and \mathcal{S} strongly commute iff

$$\begin{array}{ccc} & \rightarrow_{\mathcal{R}} & \\ \downarrow_{\mathcal{S}} & & \downarrow^*_{\mathcal{S}} \\ & \rightarrow^{\equiv}_{\mathcal{R}} & \end{array}$$

Strong Commutation

Theorem :

- If \mathcal{R} and \mathcal{S} strongly commute , then they commute.
- If \mathcal{R} and \mathcal{S} are confluent and commute, then $\mathcal{R} \cup \mathcal{S}$ is confluent.

(Famous) Example : confluence of $\beta\eta$

Example :

$$\begin{array}{l} (\lambda x.M)N \rightarrow_{\beta} M\{x/N\} \\ \lambda x.M x \rightarrow_{\eta} M \quad \text{If } x \notin \mathbf{fv}(M) \end{array}$$

Let $\mathcal{R} = \beta$ and $\mathcal{S} = \eta$. Now,

- Show that β is confluent (done).
- Show that η is confluent.
- Show that β and η strongly commute.
- Conclude that $\beta \cup \eta$ is confluent.

Confluence by interpretation

Theorem :

Let \mathcal{R} and \mathcal{S} be two relations s.t. \mathcal{R} is confluent and terminating.

If there is a relation \mathcal{T} on the set of \mathcal{R} -normal forms s.t.

1. $\rightarrow_{\mathcal{T}}^* \subseteq \rightarrow_{\mathcal{R} \cup \mathcal{S}}^*$ and
2. $a \rightarrow_{\mathcal{S}} b$ implies $\mathcal{R}(a) \rightarrow_{\mathcal{T}}^* \mathcal{R}(b)$

then if \mathcal{T} is confluent, $\mathcal{R} \cup \mathcal{S}$ is also confluent.

(Famous) Example : confluence of λx

$$\begin{aligned}(\lambda x.t) u &\rightarrow_B t[x/u] \\(t u)[x/v] &\rightarrow_{\mathbf{x}} (t[x/v] u[x/v]) \\(\lambda y.t)[x/v] &\rightarrow_{\mathbf{x}} \lambda y.t[x/v] && \text{if } x \neq y \text{ \& } y \notin FV(v) \\y[x/v] &\rightarrow_{\mathbf{x}} y && \text{if } x \neq y \\x[x/v] &\rightarrow_{\mathbf{x}} v\end{aligned}$$

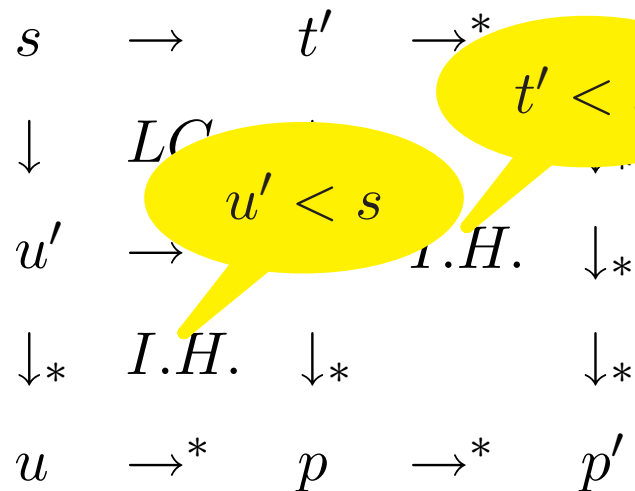
Let $\mathcal{R} = \mathbf{x}$ and $\mathcal{S} = B$ and $\mathcal{T} = \beta$, Now,

- Show that \mathbf{x} is confluent and terminating.
- Show that $\beta \subseteq (\mathbf{x} \cup B)^*$.
- Show that $a \rightarrow_B b$ implies $\mathbf{x}(a) \rightarrow_{\beta}^* \mathbf{x}(b)$.
- Since β is confluent $\lambda x = \mathbf{x} \cup B$ is confluent.

Confluence by critical pairs

Lemma $\frac{[\text{for all } y \text{ s.t. } x \rightarrow y \text{ we have } y \in SN]}{\text{locally } \forall x. x \in SN} \Rightarrow x \in SN$

Proof. (By Huet) By well-founded induction on $s \in SN$.



Important remark

The following (infinite) system on natural numbers :

$$\mathcal{R} = \left\{ \begin{array}{l} 2.n \quad \rightarrow \quad 2.n + 1 \\ 2.n \quad \rightarrow \quad a \\ 2.m + 1 \quad \rightarrow \quad 2.m + 2 \\ 2.m + 1 \quad \rightarrow \quad b \end{array} \right.$$

is **locally confluent** but not **confluent** : $a \leftarrow 0 \rightarrow^* b$

In fact it is not SN

$$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \dots$$

Towards local confluence : critical pairs

A **critical pair** between two **variable disjoint rules** $l \rightarrow r$ and $g \rightarrow d$ of \mathcal{R} (not necessarily distinct rules) is a pair $\langle \sigma(r), \sigma(l)[\sigma(d)]_p \rangle$ s.t.

1. $p \in Pos(l)$ and $l|_p$ is not a variable.
2. σ is a principal unifier of $l|_p$ and g .

Observe that

$$\begin{aligned} \sigma(r) \leftarrow \sigma(l) &= \sigma(l)[\sigma(l)|_p]_p \\ &= \sigma(l)[\sigma(l|_p)]_p \\ &= \sigma(l)[\sigma(g)]_p \rightarrow \sigma(l)[\sigma(d)]_p \end{aligned}$$

Example :

$$\mathcal{R} = \begin{cases} f(g(x), g(y), a) & \rightarrow j(x, y, a) \\ g(b) & \rightarrow b \\ h(b) & \rightarrow b \end{cases}$$

The critical pairs are :

$$j(b, y, a) \leftarrow f(g(b), g(y), a) \rightarrow f(b, g(y), a)$$

$$j(x, b, a) \leftarrow f(g(x), g(b), a) \rightarrow f(g(x), b, a)$$

Example :

$$\mathcal{R} = \left\{ \begin{array}{ll} f(f(x)) & \rightarrow g(x) \\ f(b) & \rightarrow b \\ b & \rightarrow a \end{array} \right.$$

The critical pairs are :

$$\begin{array}{llll} g(b) & \leftarrow & f(f(b)) & \rightarrow f(b) \\ g(f(x)) & \leftarrow & f(f(f(x))) & \rightarrow f(g(x)) \\ b & \leftarrow & f(b) & \rightarrow f(a) \end{array}$$

Local confluence by critical pairs

Theorem : Let \mathcal{R} be a rewrite system. Then \mathcal{R} is locally confluent iff every critical pair of \mathcal{R} is joinable.

Proof.

The *only if* implication is trivial.

For the *if* implication, let us take any case of the form

$$v \leftarrow t \rightarrow u$$

Three cases are possible :

- Disjoint reductions :

$$\begin{array}{ccccc}
 f(b, h(b), a) & \leftarrow & f(g(b), h(b), a) & \rightarrow & f(g(b), b, a) \\
 f(b, h(b), a) & \rightarrow & f(b, b, a) & \leftarrow & f(g(b), b, a)
 \end{array}$$

- Not disjoint and not critical :

$$\begin{array}{ccccc}
 j(h(b), h(b), a) & \leftarrow & f(g(h(b)), g(h(b)), a) & \rightarrow & f(g(h(b)), g(b), a) \\
 j(h(b), h(b), a) & & & & f(g(h(b)), g(b), a) \\
 \downarrow & & & & \downarrow \\
 j(b, h(b), a) & \rightarrow & j(b, b, a) & \leftarrow & f(g(b), g(b), a)
 \end{array}$$

- Not disjoint and critical : we close the diagram by the hypothesis.

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Decidable case of confluence

Theorem : Let \mathcal{R} be a **finite** and **SN** rewrite system. Then confluence of \mathcal{R} is decidable.

Proof. The algorithm :

1. Generate all the critical pairs.
2. For each critical pair $\langle u, v \rangle$, compute **arbitrary** some normal form \hat{u} of u and some normal form \hat{v} of v . If $\hat{u} \neq \hat{v}$, then **fail**.
3. Otherwise (no fail for some critical pair), **succeed**.

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Remark that

- If the algorithm fails, then there is a critical pair which is not joinable, so \mathcal{R} is not confluent by the previous theorem.
- If the algorithm succeeds, then every critical pair is joinable, so that \mathcal{R} is confluent by the previous theorem.

Ortho no duplication of variables on the left of rules

A system is **orthogonal** iff it is **left linear** and **has no critical pairs**.

Example :

$$0 + y \quad \rightarrow \quad y$$

$$s(x) + y \quad \rightarrow \quad s(x + y)$$

$$0 * y \quad \rightarrow \quad 0$$

$$s(x) * y \quad \rightarrow \quad (x * y) + y$$

Confluence by orthogonality

Theorem : If \mathcal{R} is orthogonal, then it is confluent.

Proof. We define a notion of **parallel reduction** for first-order terms :

$$\frac{}{s \gg s} \quad (\text{reflexivity}) \quad \frac{l \rightarrow r \in \mathcal{R} \text{ and } \sigma \text{ a subst.}}{\sigma(l) \gg \sigma(r)} \quad (\text{head})$$
$$\frac{s_1 \gg t_1 \dots \dots s_n \gg t_n}{f(s_1, \dots, s_n) \gg f(t_1, \dots, t_n)} \quad (\text{context})$$

Example :

$$\text{For } \mathcal{R} = \begin{cases} f(x, y) \rightarrow h(y, y) \\ a \rightarrow b \end{cases} \text{ we have } \begin{array}{l} f(f(a, c), a, f(a, c)) \\ \gg \\ f(h(a, a), b, f(b, c)) \end{array}$$

Now use the **confluence by confluence** theorem.

so it is strongly confluent

1. Observe that $\rightarrow \subseteq \gg$
2. Show that \gg has the diamond property.
3. Conclude that \rightarrow is confluent.

Important remark (I)

Left linearity alone is not sufficient for confluence.

Example :

$$\begin{array}{lll} a \rightarrow b & b \rightarrow a & c \rightarrow a \\ a \rightarrow c & b \rightarrow d & c \rightarrow e \end{array}$$

Two not joinable terms :

$$e \stackrel{*}{\leftarrow} a \stackrel{*}{\rightarrow} d$$

Important remark (II)

Absence of critical pairs is not sufficient for confluence.

Example :

$$f(x, x) \rightarrow a$$

$$f(x, g(x)) \rightarrow b$$

$$c \rightarrow g(c)$$

Two not joinable terms :

$$b \stackrel{*}{\leftarrow} f(c, c) \stackrel{*}{\rightarrow} a$$

Relaxing orthogonality

$$\text{por}(\mathbf{t}, x) \rightarrow \mathbf{t}$$

$$\text{por}(x, \mathbf{t}) \rightarrow \mathbf{t}$$

This system is not orthogonal but the critical pair is trivial.

A system \mathcal{R} is **parallel closed** iff for every critical pair $\langle u, v \rangle$ of \mathcal{R} we have $v \gg u$.

Theorem :

If \mathcal{R} is left-linear and parallel closed then it is confluent.