Topics in Computational Geometry

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4. Pseudotriangulations et graphes/complexes de visibilité

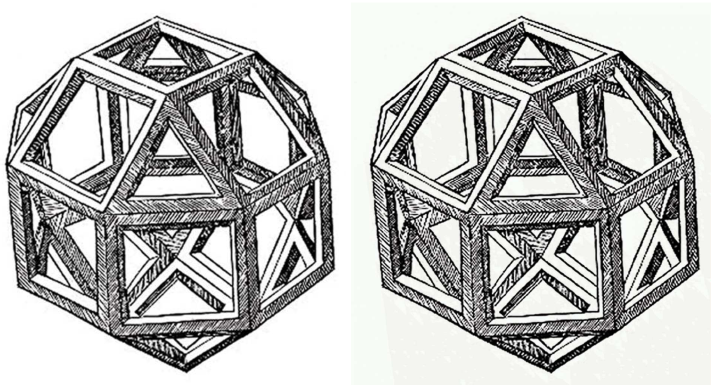
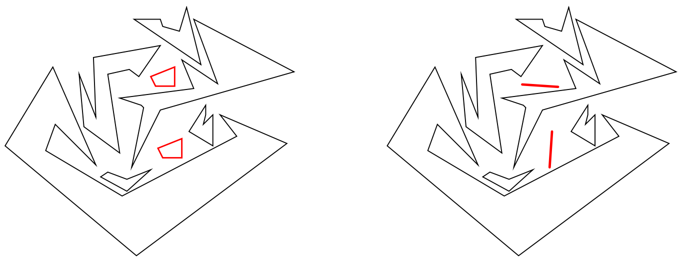
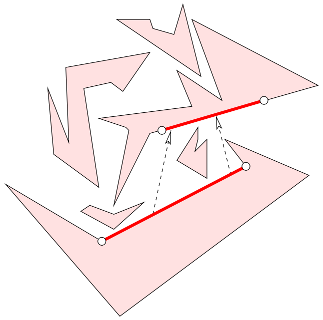


Figure 1: Première illustration d'un rhombicuboctaèdre en creux par Leonardo da Vinci pour la Divina proportione

LE PROBLÈME DU DÉMÉNAGEUR DE PIANO



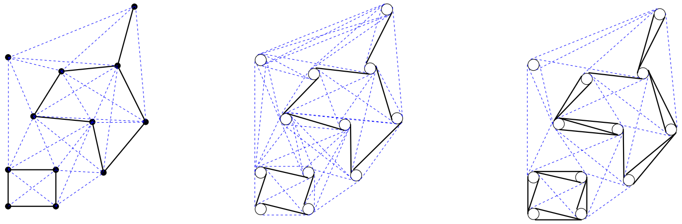
FACTEUR DE FORME



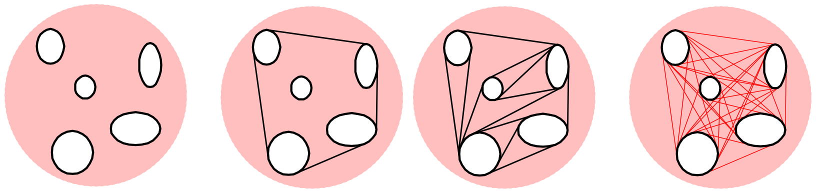
Physique. Le facteur de forme est la fraction d'energie diffuse quittant une surface et atteignant directement une autre surface.

Géométrie. Le facteur de forme F_{ij} entre les surfaces s_i et s_j est la mesure de l'ensemble X_{ij} des droites passant par deux points mutuellement visibles de s_i et s_j. Il existe une unique mesure sur l'espace des droites invariante par déplacements.

VISIBILITY GRAPH OF POLYGONAL OBSTACLES

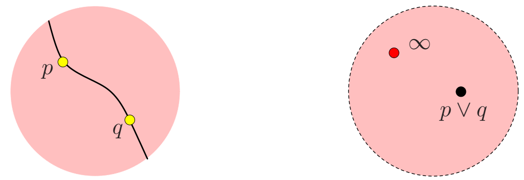


MAIN RESULT



Theorem 1 (Habert & P. 04/06, Angelier & P. 03, P. & Vegter 96). *The convex hull, a (greedy) pseudotriangulation, the visibility graph of a finite family of pairwise disjoint convex bodies of a* <u>*topological plane on* $\mathbb{R}^2$</u> *are each computable using only* $O($*size of the output*$+$ *size of the input* $\times \log$ *size of the input*$)$ *probes of its* <u>*chirotope*</u>. □

$\mathbb{R}^2$-PLANES (TOPOLOGICAL PLANES ON $\mathbb{R}^2$)



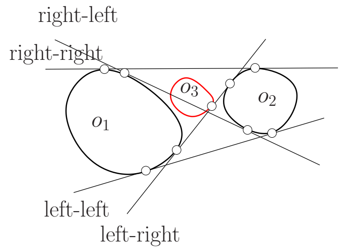
space of points $\approx \mathbb{R}^2$ space of lines $\approx \mathbb{P}^2 \setminus \{\infty\}$

Definition 1. *A pseudoline of* $\mathbb{R}^2$ *is the image of* $\mathbb{R} \times \{0\}$ *by an homeomorphism of* $\mathbb{R}^2$. □

Definition 2. *An* $\mathbb{R}^2$*-plane is a point-line incidence structure whose point set* X *is a topological space homeomorphic to* $\mathbb{R}^2$ *and whose line set satisfies the following two conditions :*
(1) Each line is a pseudoline of X*;*
(2) Any two distinct points are contained in exactly one line, their joining line. □

Definition 3. *An affine* $\mathbb{R}^2$*-plane is a* $\mathbb{R}^2$*-plane such that for every point-line pair* (p, l) *there exists a unique line* k *through* p *such that either* $k = l$ *or* k *and* l *are disjoint; the line* k *is called the parallel to* l *through* p. □

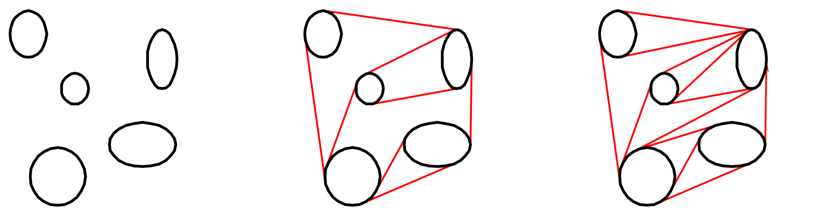
CHIROTOPES OF FINITE PLANAR FAMILIES OF PAIRWISE DISJOINT CONVEX BODIES



$(1,2,3)$	open left side	open right side	initial segment	middle segment	terminal segment
right-left	1	1	0	1	0
right-right	0	1	0	0	0
left-left	1	0	0	0	0
left-right	1	0	0	1	0

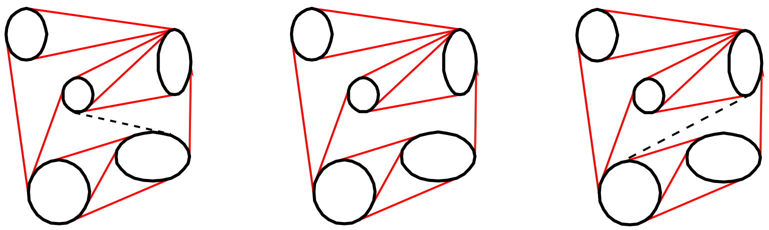
Definition 4. *The chirotope of a finite planar family of pairwise disjoint convex bodies is defined as the map that assigns to each ordered triple of indices of the family the position vector of the corresponding ordered triple of bodies, i.e., the vector of truth values (true or false) of the twenty relations «the third body of the triple intersects the open left side (open right side, initial/middle/terminal segment) of the right-left (right-right/left-left/left-right) directed double tangent joining the first body of the triple to the second body of the triple».* □

PSEUDOTRIANGULATIONS



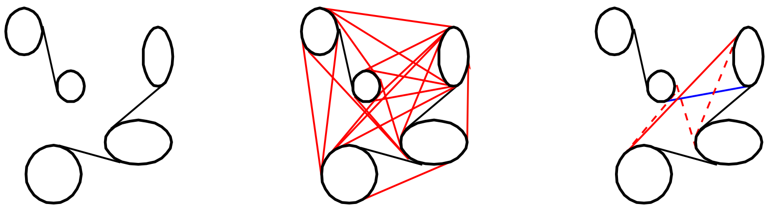
1. O a set of n pairwise disjoint convex bodies
2. B the set of $k \leq 2n(n-1)$ undirected free bitangents
3. A pseudotriangulation G is a maximal (for $\subseteq$) planar subset of B
4. $C \subset G$, $|G| = 3n - 3$
5. $(\forall G)\,(\forall b \in G \setminus C)\,(\exists!\, G' \neq G)\ \ (G \setminus G' = \{b\})$.

FLIP GRAPH OF PSEUDOTRIANGULATIONS



1. G a pseudotriangulation, $C \subset G$, $|G| = 3n - 3$
2. **[Flips]** $(\forall G)\,(\forall b \in G \setminus C)\,(\exists!\, G' \neq G)\;\; (G \setminus G' = \{b\})$ $\qquad G \xrightarrow{b} G'$.
3. **[Connectivity]** Le graphe d'adjacence des pseudotriangulations est 'fortement' connexe.
$$\begin{cases} G = G_0 \xrightarrow{b_1} G_1 \cdots \xrightarrow{b_k} G_k = G' \\ G \cap G' \subseteq G_i \qquad (i = 0, \ldots, k) \end{cases}$$
4. $\{H \mid C \subseteq H, H \text{ planaire}\}$ ordonné par inclusion est un polytope abstrait de dimension $3n - 3 - |C|$.

GREEDY PSEUDOTRIANGULATIONS



1. $\kappa \subset \mathcal{B}$ planar
2. $\mathcal{B}_\kappa = \{v \in \mathcal{B} \mid \kappa \cup \{v\} \text{ is planar}\}$, $\Theta(v)$, $\mathbb{B}_\kappa = \mathcal{B}_\kappa \times \mathbb{Z}$, $\Theta(v, k) = \Theta(v) + k\pi$
3. We define a partial order $\prec_\kappa$ on $\mathbb{B}_\kappa$ as follows : $v \prec_\kappa v'$ if there exists a sequence of bitangents $v = v_1, v_2, \ldots, v_k = v'$ such that v_i and v_{i+1} touch the same directed disk and $\Theta(v_i) < \Theta(v_{i+1})$

   Theorem 2 (PV96,AP03)**.** *Two crossing bitangents of $\mathbb{B}_\kappa$ are comparable with respect to $\prec_\kappa$ and the operator $\varphi(.;\kappa) : \mathbb{B}_\kappa \to \mathbb{B}_\kappa$ that associates with $v \in \mathbb{B}_\kappa$ the minimum element of the set of $u \in \mathbb{B}_\kappa$ such that u crosses v and $v \prec_\kappa u$ is well-defined, one-to-one, and onto.* □

GREEDY FLIP PROPERTY

For any filter I of the poset $(\mathbb{B}_\kappa, \prec_\kappa)$, we define a maximal planar set of bitangents $\mathrm{G}(I) = \{v_1, v_2, \ldots, v_k\}$ recursively: v_1 is minimal in I, and, for $i > 1$, v_{i+1} is minimal in the set of bitangents $v \in I \setminus \{v_1, \ldots, v_i\}$ such that $\kappa \cup \{v_1, \ldots, v_i, v\}$ is planar—note that this set is empty after $3n - 3$ steps.

Theorem 3 (PV96,AP03)**.** *Let v be a minimal element of the filter I of the poset $(\mathbb{B}_\kappa, \prec_\kappa)$. Then*

1. $\mathrm{G}(I) = I \setminus \varphi(I;\kappa)$;
2. $\mathrm{G}(I \setminus \{v\})$ *is obtained from* $\mathrm{G}(I)$ *by flipping* v. □

ENUMERATION OF PSEUDOTRIANGULATIONS

Let I be a filter (for example the filter of bitangents with positive angle) of $(\mathbb{B}_\kappa, \prec_\kappa)$, and let $G_0 = G(I)$ be the associated greedy pseudotriangulation. For $H = (F, G)$, $F \cap G = \emptyset$, $\kappa \subset F \cup G \subset I$ a pseudotriangulation, we define $\omega(H)$ to be the pair $(F \cup \{v\}, G \setminus \{v\})$ and $\phi(H)$ to be the pair $(F, G \setminus \{v\} \cup \{v'\})$ where $v = \min_{<_\kappa} G$ and v' is the shadow of v in $F \cup G$. (Observe that ω and ϕ are defined only if $G \neq \emptyset$.) For $G \supseteq \kappa$ a pseudotriangulation of the o_i's we define a sequence $H(G) = H_1, H_2, \ldots, H_i = (F_i, G_i)$ by setting $H_0 = (\emptyset, G_0)$, and

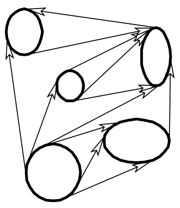
$$H_{i+1} = \begin{cases} \omega(H_i), & \text{if } \min_{<_\kappa} G_i \in G; \\ \phi(H_i), & \text{otherwise.} \end{cases} \tag{1}$$

Il est aisé de voir that the sequence $H(G)$ is finite, that its last term is $\overline{H}(G) = (G, \emptyset)$, and that $H(G) = H(G')$ iff. $G = G'$. This proves the strong connectivity of the flip graph of pseudotriangulations and, with some extra work, provides a linear space traversal algorithm of the flip graph whose complexity is linear amortized per node

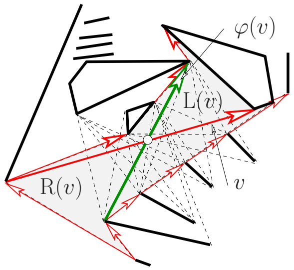
VISIBILITY GRAPH ALGORITHM

Algorithm Greedy Flip Algorithm

compute the greedy pseudo-triangulation $G := \mathrm{G}(I(0))$;
repeat
    select a minimal bitangent v in G with angle less than π;
    flip v; (i.e., replace v by $\varphi(v)$)
until there are no more bitangents of angle less than π.



$\mathrm{G}(S)$



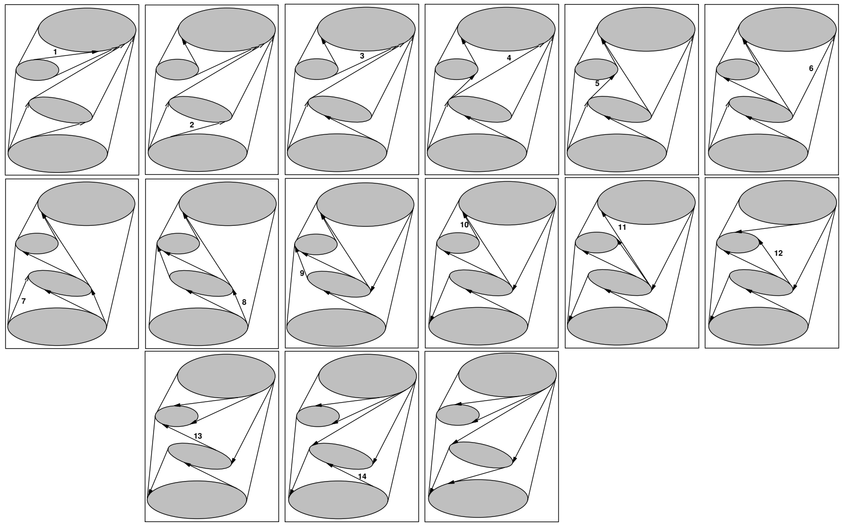
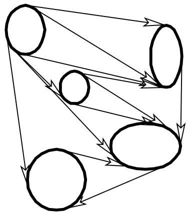


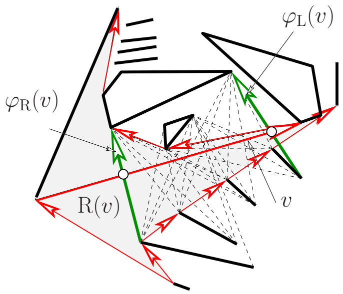
Figure 2: The greedy flip algorithm. At each step the internal bitangent of minimal slope in the current pseudo-triangulation is flipped

THE FLIP OPERATION (VERSION AP03)

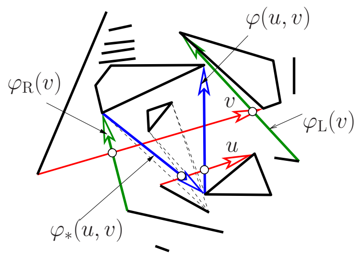
Theorem 4 (AP03)**.** *The flip operation of the greedy flip algorithm can be implemented in constant amortized time using only the chirotope of the convex bodies. Furthermore the only necessary data structures are linked structures to represent the incidence relations in the sweep structure S and its associated pseudotriangulations $\mathrm{G}(S)$ and $\mathrm{G}_*(S)$, a list to store the minimal bitangents of $\mathrm{G}(S)$, and for each minimal bitangent v pointers to the arcs of $\mathrm{G}(S)$ that $\varphi_{\mathrm{R}}(v)$ and $\varphi_{\mathrm{L}}(v)$ leave and pointers to the arcs of $\mathrm{G}_*(S)$ that $\varphi_{\mathrm{R}}(v)$ and $\varphi_{\mathrm{L}}(v)$ enter.* □

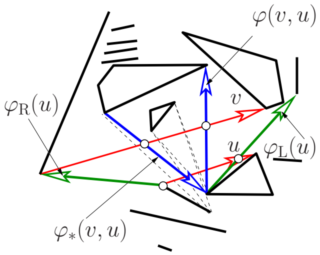


$\mathrm{G}_*(S)$



"SUM OF SQUARES" THEOREM

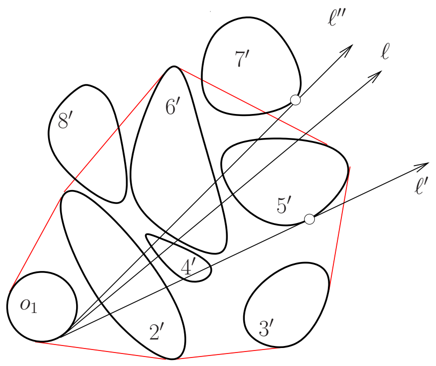




$\varphi(u, v) = \varphi(u; \{\varphi_{\mathrm{R}}(v), \varphi_{\mathrm{L}}(v)\})$.

Theorem 5 (AP03)**.** *The number of pairs of free bitangents (u, v) such that $\varphi(u, v) = \varphi(v, u)$ and $\varphi_*(u, v) = \varphi_*(v, u)$ is bounded by a constant times the number of free bitangents.* □

INITIALIZATION OF THE GFA



Theorem 6 (HP03)**.** *There is a* G(I) *computable in* $O(n \log n)$ *time using only the chirotope of the convex bodies.* □

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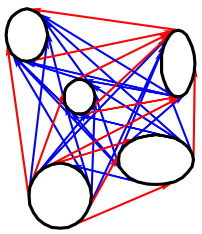
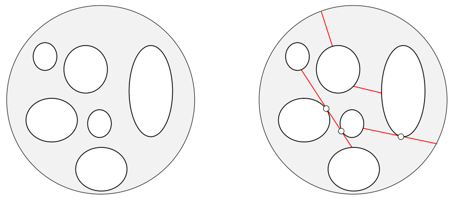
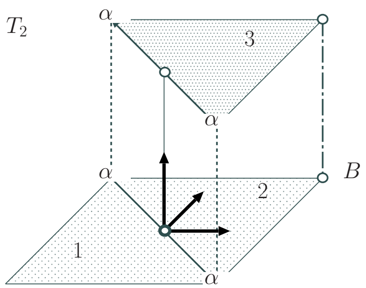
COMPLEXES DE VISIBILITÉ

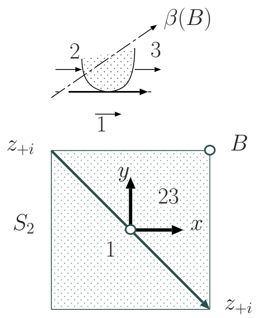


1. Une collection de n corps convexes : $O_i \subset \mathbb{R}^2$
2. Espace libre : $\mathbb{F} = \mathbb{R}^2 \setminus \bigcup_i \text{Interior}(O_i)$
3. Espace des droites de $\mathbb{F}$: $\mathbb{V}$
4.

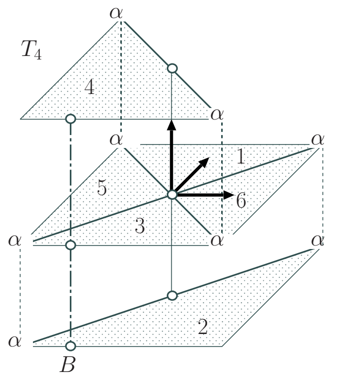


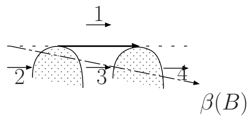
TOPOLOGY OF THE VISIBILITY COMPLEX

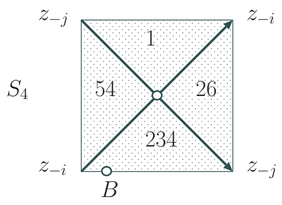




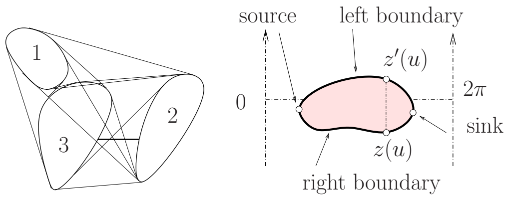
TOPOLOGY OF THE VISIBILITY COMPLEX







BORDS GAUCHE ET DROIT D'UNE 2-CELLULE



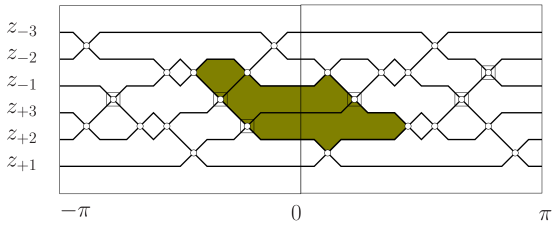
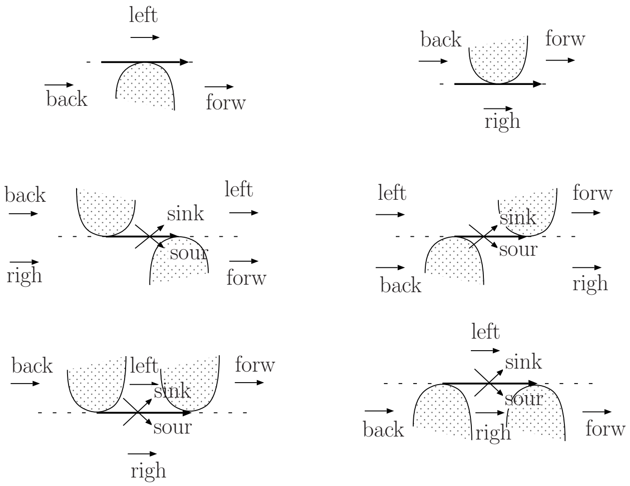


Figure 3: Example of arrangement of curves $z_{\epsilon o}$'s for a collection of three convex bodies : each curve $z_{\epsilon o}$ is stretched and bended so that it stays roughly parallel to the equator of the space of directed lines. There is a one-to-one and onto correspondence between the set of vertices of the arrangement and the set of free and non-free directed bitangents. The (6) vertices marked with a square correspond to non-free bitangents. The dashed region represents the projection in the space of directed lines of the 2-cell containing the elements of the visibility complex with backward view the disk labeled 3 and forward view the disk labeled 2.

INCIDENCE RELATIONS



INCIDENCE OPERATORS

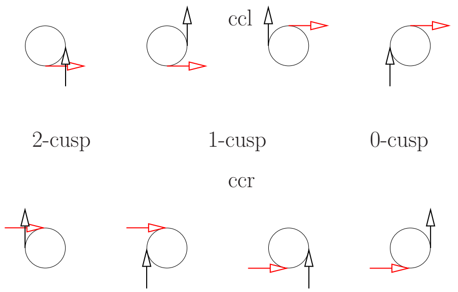


Figure 4: Regular, cusp and double cusp arcs.

1. Sink and source of a 1- or 2-cell;
2. ccr and ccl operators;
3. For $\sigma \in X_0 \cup X_1$ and $\alpha \in \{\text{left}, \text{back}, \text{righ}, \text{forw}, \text{sink}\}$ we set (if defined)

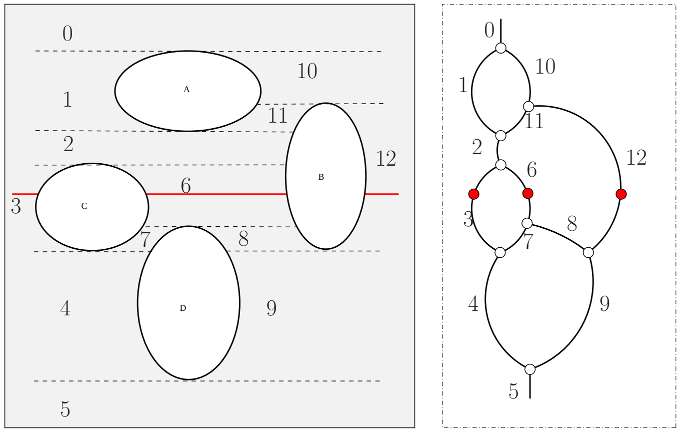
$$\varphi_\alpha(\sigma) = \text{sink}(\sigma_\alpha), \qquad \alpha \in \{\text{left}, \text{back}, \text{righ}, \text{forw}, \text{sink}\}. \tag{2}$$

COMPUTING THE RIGHT BOUNDARY OF A FACE

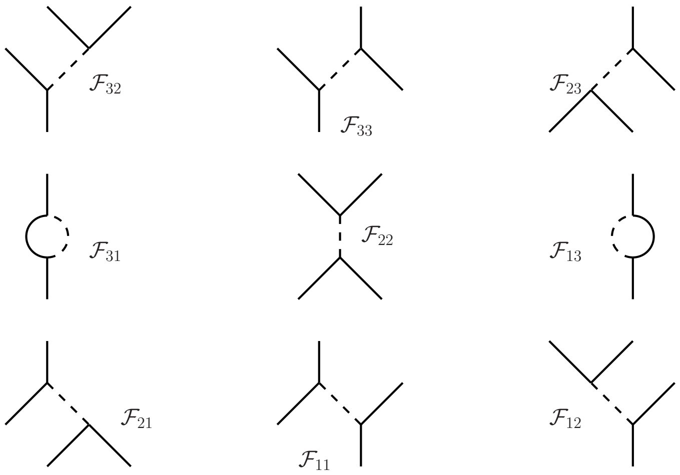
Here is an algorithm to compute the right boundary of the face σ whose source is b.
TO BE CHECKED

1 push(b); $b_2 \leftarrow$ ccr(b);
2 **if** (b, b_2) 2-cusp **then goto 6**
3 **while**(b, b_2) 0-cusp {
push(b_2); /* b_2 IS 'FAT' */
$b_2 \leftarrow$ ccl(b_2);
}
4 **do** {
push (b_2); /*b_2 IS 'REGULAR' */
$(b_1, b_2) \leftarrow (b_2, \text{ccl}(b_2))$;
} **while** (b_1, b_2) 0-cusp
5 **while** (b_1, b_2) 2-cusp {
push (b_2); /* b_2 IS 'FAT'*/
$b_2 \leftarrow$ ccr(b_2);
}
6 push(b_2); /*b_2 IS THE SINK OF σ, I.E., $b_2 = \varphi_{\text{sink}}(b)$ */
exit;

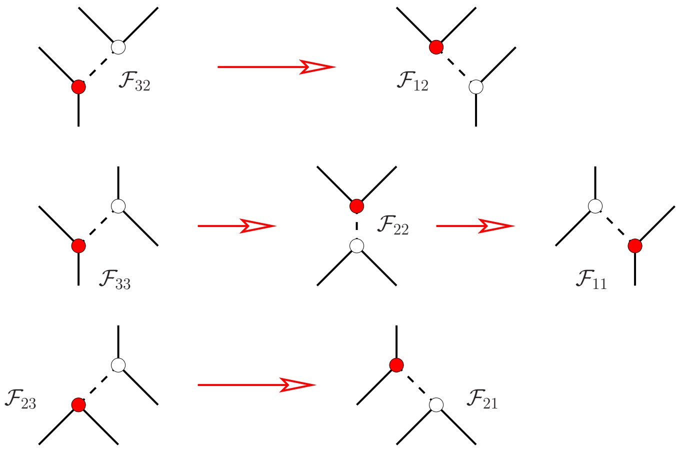
CROSS-SECTIONS OF THE VISIBILITY COMPLEX



TYPOLOGY OF THE 2-FACES OF A CROSS-SECTION



TRANSITIONS : $\mathcal{F}_{i+1,j+1} \to \mathcal{F}_{j,i}$



BALAYAGE DROIT

1. Fonctions supports des $\{o_i\}$

$$\theta \in \mathbb{R} \;\mapsto\; z_{+o}(\theta) = \inf_{p \in o} \langle p, \imath \exp(\imath\theta) \rangle \in \mathbb{R}$$
$$\theta \in \mathbb{R} \;\mapsto\; z_{-o}(\theta) = \sup_{p \in o} \langle p, \imath \exp(\imath\theta) \rangle \in \mathbb{R}$$

2. $z_{-o}(\theta + \pi) = -z_{+o}(\theta)$

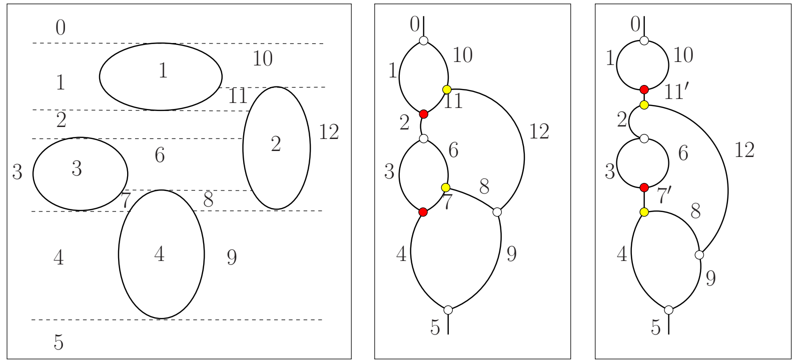
Theorem 7. *Soit $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_k < \pi$ la suite croissante des angles des bitangentes libres orientées vers le haut et pour toute arête σ de la section $S_i \approx S(\theta)$ pour $\theta \in (\theta_{i-1}, \theta_i)$ du complexe de visibilite soit* death(σ) *la plus petite solution supérieure à θ_{i-1} de l'équation $z_{\sigma^+}(\theta) = z_{\sigma^-}(\theta)$. Alors θ_i est le plus petit des* death(σ) *pour σ arête de S_i.* □

Theorem 8. *Les k bitangentes libres d'une collection de n corps convexes disjoint deux à deux d'un plan topologique affine sont calculables en temps $O(k \log n)$ et espace de travail linéaire.* □

EXAMPLE

$2 \in \mathcal{F}_{22}$, $9 \in \mathcal{F}_{32}$, $7, 11 \in \mathcal{F}_{33}$, $12 \in \mathcal{F}_{31}$, $1, 3 \in \mathcal{F}_{13}$, $6, 10 \in \mathcal{F}_{21}$, $8 \in \mathcal{F}_{11}$, $4 \in \mathcal{F}_{12}$, $5 = \hat{0}$, $0 = \hat{1}$. death$(2) = v_{-3,1}$, death$(11) = v_{1,-2}$, death$(9) = v_{4,2}$, death$(7) = v_{3,-4}$.

$$\text{death}(7) = \text{death}(11) < \text{death}(2) < \text{death}(9)$$



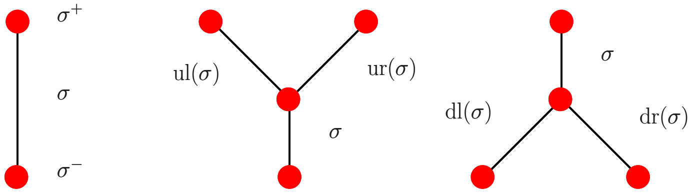
PARTIAL ORDER – TOPOLOGICAL CROSS-SECTIONS

$$\begin{tikzcd} \mathbb{W} \arrow[r, "q'"] \arrow[d, "p"'] & \hat{\mathbb{L}} \arrow[d, "p'"] \\ \mathbb{V} \arrow[r, "q"] & \mathbb{L} \end{tikzcd}$$

1. X_0, X_1, X_2 denote the sets of vertices, edges, and faces of the visibility complex $\mathbb{W}$
2. Partial order $\prec$ defined via the covering relations : $\mathrm{sour}(\sigma) \prec \sigma \prec \mathrm{sink}(\sigma)$

TOPOLOGICAL CROSS-SECTIONS

1. proper filter of $(X_0, \prec)$ $\quad I \leftrightarrow \mathrm{S}(I) = \mathcal{E}(I) \cup \mathcal{F}(I)$ $\quad$ maximal antichains of $(X_{\geq 1}, \prec)$
2. The planar directed graph $\Gamma(I)$ induced on $\mathrm{S}(I)$ by the incidence relations.
3. For $e \in X_1$ supported by a z_{+i} one has $\mathrm{sink}(e) = \mathrm{sink}(e_{\mathrm{righ}})$ or $\mathrm{sink}(e_{\mathrm{forw}})$;
4. Let $\sigma \in \mathcal{F}(I)$. $\mathrm{sink}(\sigma)$ is minimal in I iff. $\mathrm{sink}(\sigma) = \mathrm{sink}(\sigma^+) = \mathrm{sink}(\sigma^-)$ iff. $\mathrm{sink}(\sigma^+) = \mathrm{sink}(\sigma^-)$



FLIP PROPERTY

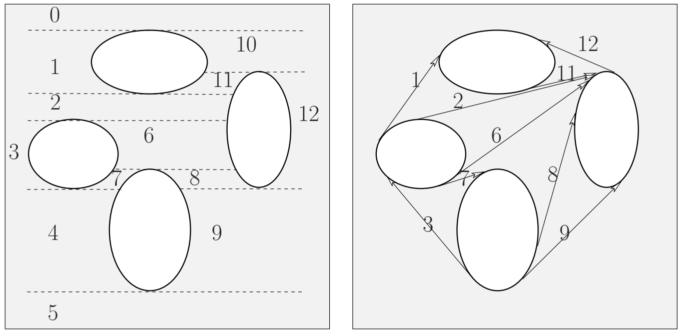


Figure 5: (Left) The horizontal map of the set of disks is a representation of the cross-section associated with the filter of vertices with positive angles : each strip-shaped region corresponds to the subset of elements with angles zero of a 2-cell of the cross-section. (Right) The horizontal greedy pseudotriangulation : the bitangent labeled i 'is' the sink of the 2-cell labeled i.

Theorem 9 (PV96,AP03)**.** $\mathrm{G}(I) = \mathrm{sink}(\mathrm{S}^\circ(I)) = \mathrm{sink}(\mathcal{F}^\circ(I))$. □

NOTES