ARRANGEMENTS DE DROITES

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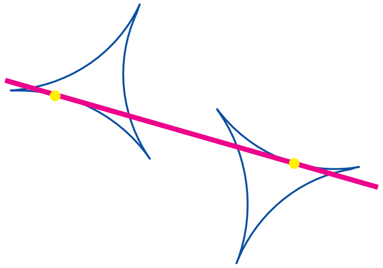
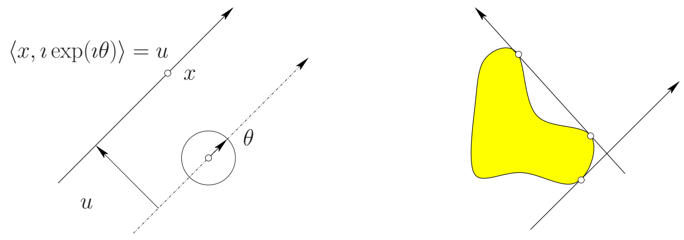


Paris
Mars 2006

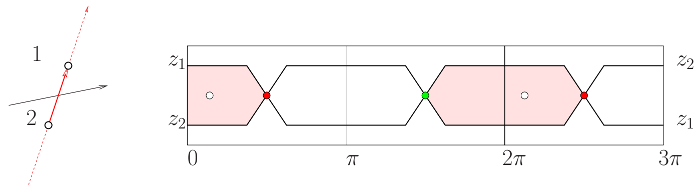
PLAN

SPACES OF LINES – SUPPORT FUNCTIONS



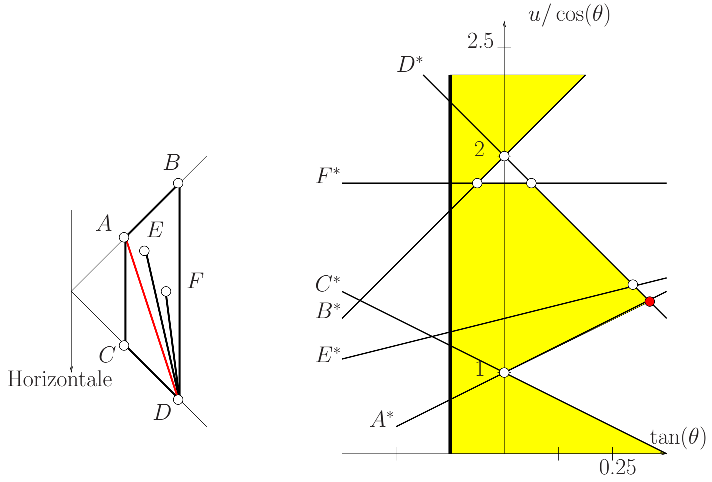
1. Let $\iota : \mathbb{R}^2 \to \mathbb{R}^2$ be the map that assigns to the pair (θ, u) the pair $(\theta + \pi, -u)$.
2. The space of directed lines is denoted $\mathbb{D}$ and is homeomorphic the quotient of $\mathbb{R}^2$ under the equivalence relation $x \sim \iota^2(x)$ via the map that assigns to (θ, u) the line with equation $\langle x, \exp(\imath(\theta + \pi/2))\rangle = u$ and direction $\exp(\imath\theta)$. The space of (undirected) lines is denoted $\mathbb{M}$ and is homeomorphic to the quotient of $\mathbb{R}^2$ under the equivalence relation $x \sim \iota(x)$ via the map that assigns to (θ, u) the line with equation $\langle x, \exp(\imath(\theta + \pi/2))\rangle = u$.
3. The support function of $X \subseteq \mathbb{R}^2$ is the map $\lambda_X : \theta \in \mathbb{R} \to \inf_{x\in X}\langle x, \imath \exp(\imath\theta)\rangle \in \mathbb{R}$ and the lines $(\theta, \lambda_X(\theta))$ are called the supporting lines of X

DUAL D'UN POINT



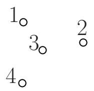
1. Le dual d'un point est l'ensemble des droites passant par ce point.
2. The support function of the point $M(x, y)$ is the map
$$u_M : \theta \in \mathbb{R} \mapsto -x \sin\theta + y \cos\theta \in \mathbb{R}$$
3. If we restrict our attention to the directions in the interval $(-\pi/2, +\pi/2)$ we might prefer the parametrization :
$$t \in \mathbb{R} \mapsto \frac{z_M(\arctan t)}{\cos \arctan(t)} = -xt + y \in \mathbb{R}$$

THE ZONE OR HORIZON THEOREM



Theorem 1 (CGL85). *The number of vertices, edges and faces incident to the faces of an arrangement of n lines cut by a $n+1$-th line is a $O(n)$.* □

BALAYAGE D'UN ARRANGEMENT DE DROITES



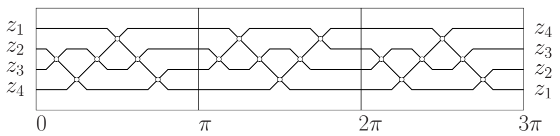


Figure 1: The restriction to the domain $[0, 3\pi] \times \mathbb{R}$ of the arrangement of the z_c's for four points : each curve z_c is stretched and bended so that it stays roughly parallel to the equator of the space of lines.

1. $\{p_i\}$ une collection de n points en position générale;
2. $\theta \in \mathbb{R} \mapsto z_i(\theta) = \langle p_i, \exp(\imath(\theta + \pi/2)) \rangle \in \mathbb{R}$; $\qquad z_i(\theta + \pi) = -z_i(\theta)$;
3. X_0, X_1, X_2 the sets of 0-, 1-, and 2-cells of the arrangement of the z_i's; incidence relations; source and sink of a 1- or 2-cell; left and right boundaries of a 2-cell; partial order $\prec$ defined by the covering relations

$$\mathrm{sour}(\sigma) \prec \sigma \prec \mathrm{sink}(\sigma), \qquad \sigma \in X_1 \cup X_2. \tag{1}$$

COUNTERCLOCKWISE OPERATORS

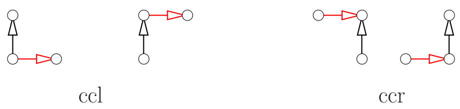


Figure 2: 0-cusp and 1-cusp arcs.

1. The ccr and ccl operators;
2. 0-cusp and 1-cusp edges;

COMPUTING THE RIGHT BOUNDARY OF A FACE

Here is an algorithm to compute the right boundary of the face σ whose source is b.
TO BE CHECKED

1 push(b); $b_2 \leftarrow$ ccr(b);
2 **if** (b, b_2) 0-cusp **then goto 6**
4 **do** {
push (b_2);
$(b_1, b_2) \leftarrow (b_2, \mathrm{ccl}(b_2))$;
} **while** (b_1, b_2) 0-cusp
6 push(b_2); /*b_2 IS THE SINK OF σ */
exit;

FILTERS AND MAXIMAL ANTICHAINS (CUTS)

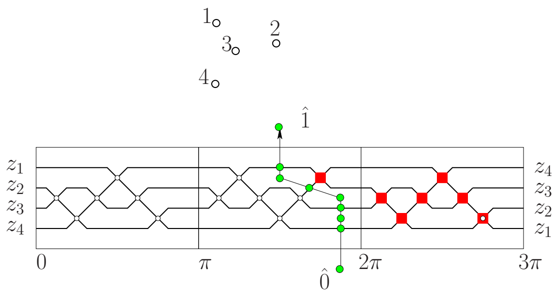
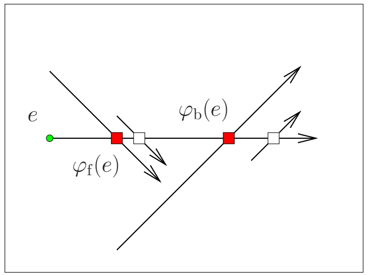
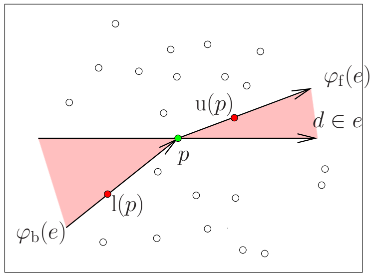


Figure 3: A filter and its associated (maximal) antichain.

1. proper filter of $(X_0, \prec)$ $\quad I \longleftrightarrow \mathrm{S}(I) = \mathcal{E}(I) \cup \mathcal{F}(I)$ $\quad$ maximal antichain of $(X_{\geq 1}, \prec)$
2. linear structure $\hat{0} e_1 \sigma_1 e_2 \sigma_2 \dots e_n \hat{1}$ on $\mathrm{S}(I)$ induced by the incidence relations on X;
3. For $e \in X_1$ one has $\mathrm{sink}(e) = \mathrm{sink}(e_\mathrm{r})$ or $\mathrm{sink}(e_\mathrm{l})$;
4. Let $\sigma \in \mathcal{F}(I)$. $\mathrm{sink}(\sigma)$ is minimal in I iff. $\mathrm{sink}(\sigma) = \mathrm{sink}(\sigma^+) = \mathrm{sink}(\sigma^-)$ iff. $\mathrm{sink}(\sigma^+) = \mathrm{sink}(\sigma^-)$

HORIZON OPERATORS





1. For $e \in X_1$ supported by z_i we define $\varphi_\mathrm{f}(e)$ as the minimal vertex $v \succ e$ leaving p_i;
2. For $e \in X_1$ supported by z_i we define $\varphi_\mathrm{b}(e)$ as the minimal vertex $v \succ e$ entering p_i;
3. $\mathrm{sink}(e) = \min_\prec\{\varphi_\mathrm{f}(e), \varphi_\mathrm{b}(e)\}$;

HORIZON TREES

Theorem 2 (Rephrasing of EG89). *Let I be a filter of X_0 and let $E = e_1 e_2 \dots e_n$ be its associated edge-cut where e_i is supported by the dual curve of the point q_i. For $i \neq n$ let $\mathrm{u}(q_i)$ be the point that $\varphi_\mathrm{f}(e_i)$ enters and for $i \neq 1$ let $\mathrm{l}(q_i)$ be the point that $\varphi_\mathrm{b}(e_i)$ leaves. Then*

1. *q_{i+1} and q_n are vertices of the convex hull C_{i+1} of the points $q_{i+1}, \dots, q_n$ ($i \geq 0$);*
2. *$q_i \notin C_{i+1}$ and the vertex q_{i+1} of C_{i+1} is "visible" from q_i, i.e. $C_{i+1} \cap [q_i q_{i+1}] = \{q_{i+1}\}$;*
3. *The sequence $q_{i+1}, \mathrm{u}(q_{i+1}), \mathrm{u}^2(q_{i+1}), \dots$ is finite, its last term is q_n, and coincides with the sequence of vertices of C_{i+1} lying between q_{i+1} and q_n for the counterclockwise orientation of the boundary of C_{i+1}; in particular $\{(q_k, \mathrm{u}(q_k))\}_{k \neq n}$ is a plane tree whose root is q_n;*
4. *$\mathrm{u}(q_i)$ is the first element z of the sequence $z' = q_{i+1}, \mathrm{u}(z'), \mathrm{u}^2(z'), \dots, q_n$ such that $q_i, z, \mathrm{u}(z)$ is counterclockwise.*
5. *A similar result holds for the operator l.* □

HORIZON TREES

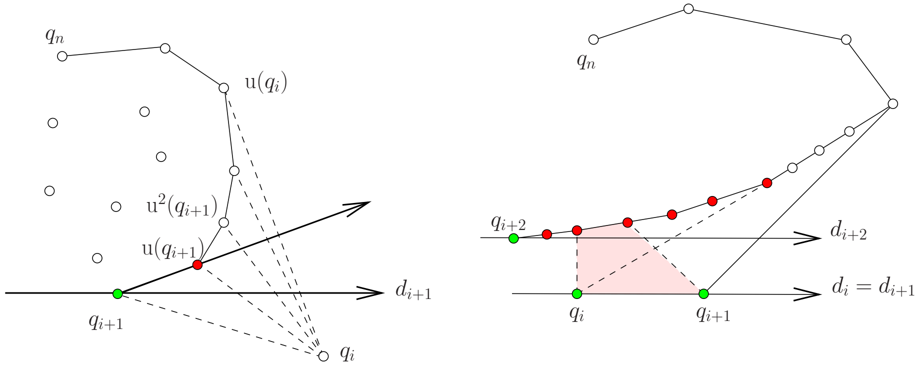
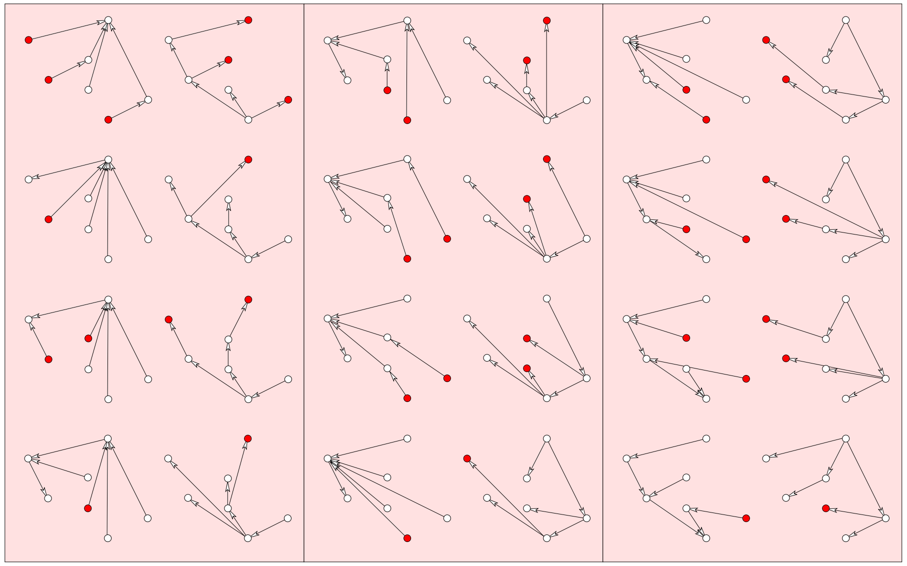
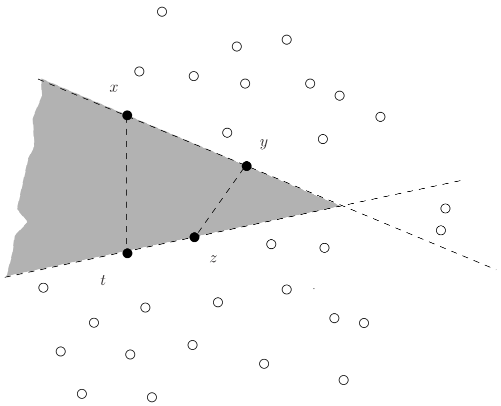


Figure 4: Blabla.

HORIZON TREES

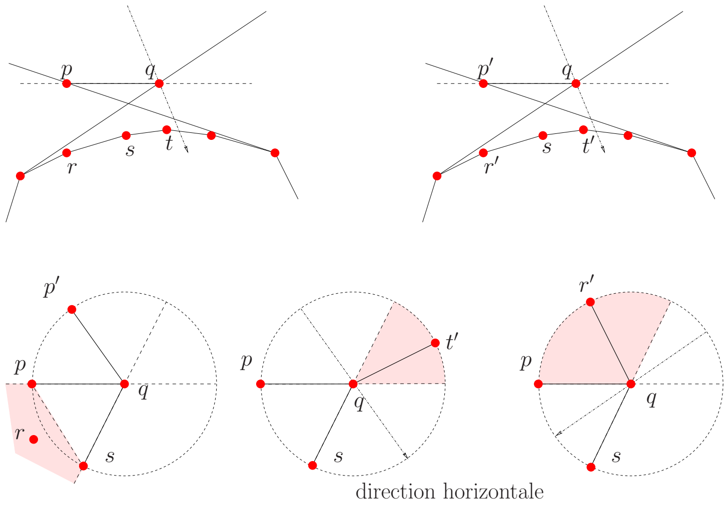


COMPLEXITY ANALYSIS



Theorem 3. *The number of 4-uplets xyzt of points of a set P of n points in general position such that (1-) x, y, z, t are in convex position and appear in this order when traversing clockwise their convex hull, and (2-) no point of P lies in the intersection of the right half-plane of the line going from x to y and the right half-plane of the line going from z to t, is a* $O(n^2)$. □

PROOF



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