Arrangements de droites

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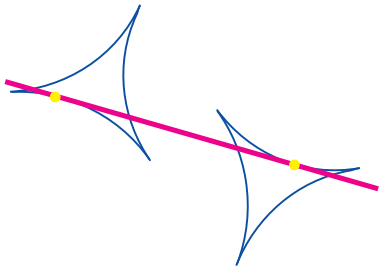
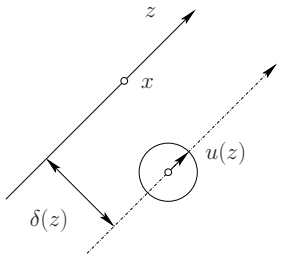


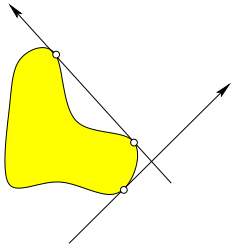
Paris
Mars 205

Plan

- Dualité point-droite
- Arrangements de droites
- Théorème de la zone
- Balayage d'un arrangement de droites
- Terminologie : complexes cellulaires, ensembles partiellement ordonnés

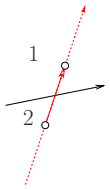
Space of directed lines – Support functions

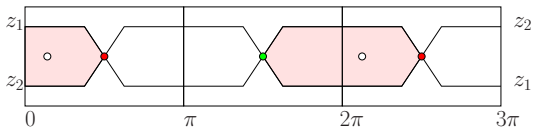




1. A directed line z is a line with a direction $u(z) \in \mathbb{S}^1 = \exp(\imath\mathbb{R}/2\pi\mathbb{Z})$ assigned to it and indicated in the figures by an arrow. The direction-coordinate of z is its direction $u(z)$ and its distance-coordinate is the real $\delta(z)$ defined as the common value of the scalar products $\langle x, \imath u\rangle$ where x ranges over the undirected version of the line. The space of directed line is denoted $\mathbb{D}$ and is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ via $z \mapsto (u(z), \delta(z))$
2. The support function of $X \subseteq \mathbb{R}^2$ is the map $\lambda_X : u \in \mathbb{S}^1 \to \inf_{x\in X}\langle x, \imath u\rangle \in \mathbb{R}$ and the directed lines $(u, \lambda_X(u))$ are called the supporting lines of X

Dual d'un point





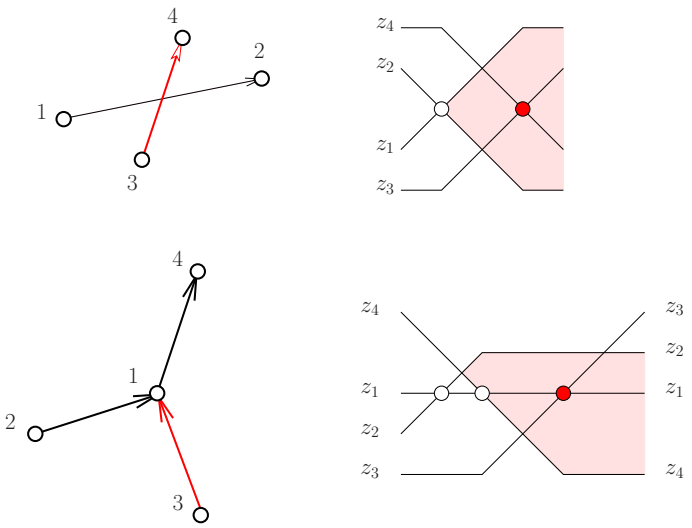
1. Le dual d'un point est l'ensemble des droites orientées passant par ce point.
2. The support function of the point $M(x, y)$ is the map

$$z_M : \theta \in \mathbb{R}/2\pi\mathbb{Z} \mapsto -x\sin\theta + y\cos\theta \in \mathbb{R}$$

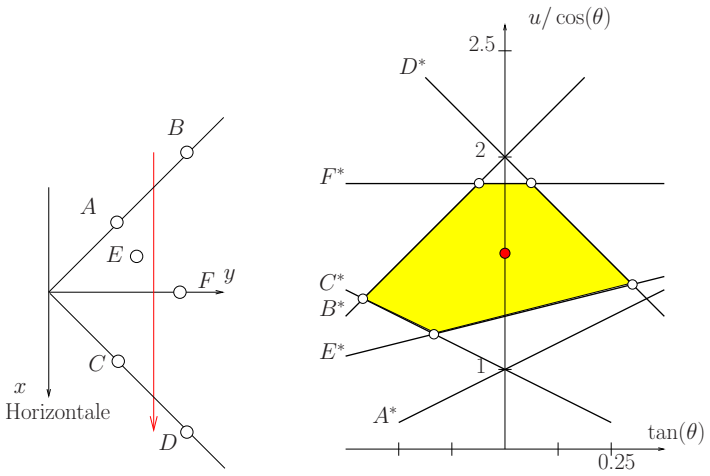
3. If we restrict our attention to the directions in the interval $(-\pi/2, +\pi/2)$ we might prefer the parametrization :

$$t \in \mathbb{R} \mapsto \frac{z_M(\arctan t)}{\cos\arctan(t)} = -xt + y \in \mathbb{R}$$

Dual d'un point



Arrangement de droites



Counterclockwise operators

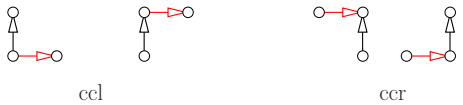


Figure 1: 0-cusp and 1-cusp arcs.

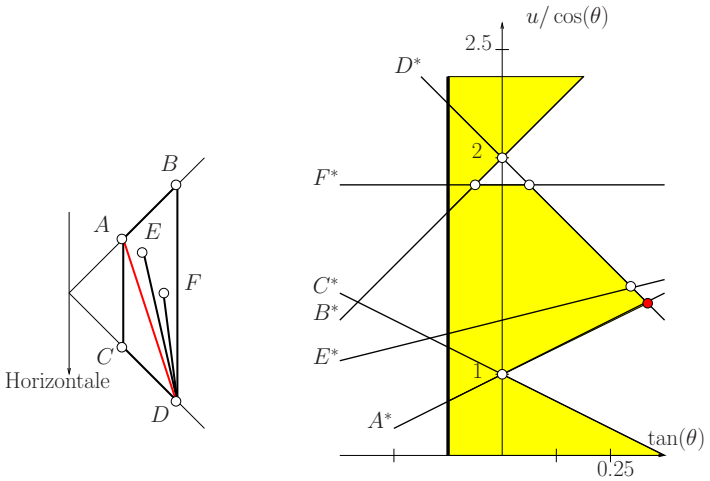
1. The ccr and ccl operators;
2. 0-cusp and 1-cusp edges;

Computing the right boundary of a face

Here is an algorithm to compute the right boundary of the face σ whose source is b.
TO BE CHECKED

1 push(b); $b_2 \leftarrow$ ccr(b);
2 **if** (b, b_2) 0-cusp **then goto 6**
4 **do** {
push (b_2);
$(b_1, b_2) \leftarrow (b_2, \text{ccl}(b_2))$;
} **while** (b_1, b_2) 0-cusp
6 push(b_2); /*b_2 is the sink of σ */
exit;

The Zone or Horizon Theorem



Theorem 1 (CGL85). *The number of vertices, edges and faces incident to the faces of an arrangement of n lines cut by a $n+1$-th line is a $O(n)$.* □

Balayage d'un arrangement de droites

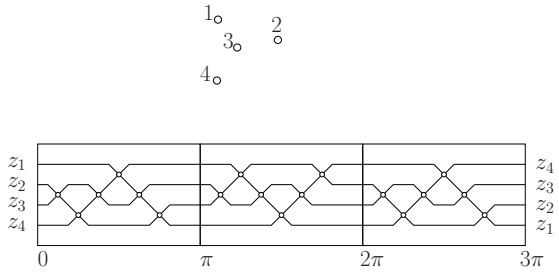


Figure 2: The restriction to the domain $[0,3\pi]\times\mathbb{R}$ of the arrangement of the z_c's for four points : each curve z_c is stretched and bended so that it stays roughly parallel to the equator of the space of lines.

1. $\{p_i\}$ une collection de n points en position générale;
2. $\theta\in\mathbb{R}\mapsto z_i(\theta)=\langle p_i,\exp(\imath(\theta+\pi/2))\rangle\in\mathbb{R}$; $z_i(\theta+\pi)=-z_i(\theta)$;
3. X_0,X_1,X_2 the sets of 0-, 1-, and 2-cells of the induced arrangement; incidence relations; source and sink of a 1- or 2-cell; left and right boundaries of a 2-cell; partial order $\prec$ defined by the covering relations

$$\mathrm{sour}(\sigma)\prec\sigma\prec\mathrm{sink}(\sigma),\qquad \sigma\in X_1\cup X_2. \tag{1}$$

Filters and maximal antichains (cuts)

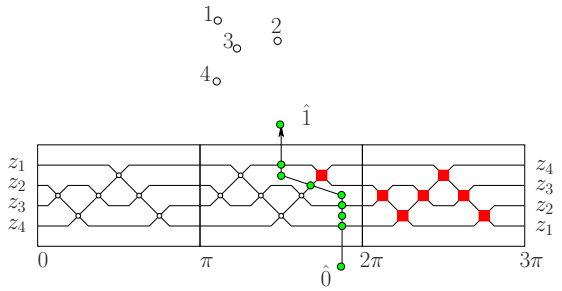
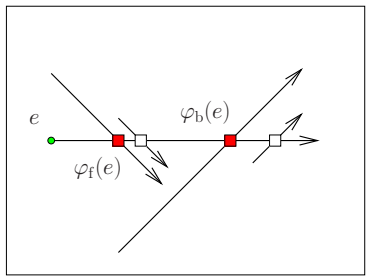
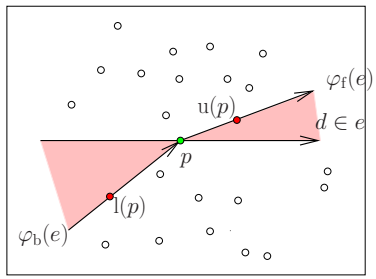


Figure 3: A filter and its associated (maximal) antichain.

1. proper filter of $(X_0,\prec)$ $\quad I\longleftrightarrow \mathrm{S}(I)=\mathcal{E}(I)\cup\mathcal{F}(I)$ $\quad$ maximal antichain of $(X_{\geq 1},\prec)$
2. linear structure $\hat{0}e_1\sigma_1e_2\sigma_2\dots e_n\hat{1}$ on $\mathrm{S}(I)$ induced by the incidence relations on X;
3. For $e\in X_1$ one has $\mathrm{sink}(e)=\mathrm{sink}(e_{\mathrm{r}})$ or $\mathrm{sink}(e_{\mathrm{l}})$;
4. Let $\sigma\in\mathcal{F}(I)$. $\mathrm{sink}(\sigma)$ is minimal in I iff. $\mathrm{sink}(\sigma)=\mathrm{sink}(\sigma^+)=\mathrm{sink}(\sigma^-)$ iff. $\mathrm{sink}(\sigma^+)=\mathrm{sink}(\sigma^-)$

Horizon operators





1. For $e\in X_1$ supported by z_i we define $\varphi_{\mathrm{f}}(e)$ as the minimal vertex $v\succ e$ leaving p_i;
2. For $e\in X_1$ supported by z_i we define $\varphi_{\mathrm{b}}(e)$ as the minimal vertex $v\succ e$ entering p_i;
3. $\mathrm{sink}(e)=\min_{\prec}\{\varphi_{\mathrm{f}}(e),\varphi_{\mathrm{b}}(e)\}$;

Horizon trees

Theorem 2 (Rephrasing of EG89). *Let J be a filter of Z^0 and let $E = e_1 e_2 \dots e_n$ be its associated edge-cut where e_i is supported by the dual curve of the point q_i. For $i \neq n$ let $\mathrm{u}(q_i)$ be the point that $\varphi_{\mathrm{f}}(e_i)$ enters and for $i \neq 1$ let $\mathrm{l}(q_i)$ be the point that $\varphi_{\mathrm{b}}(e_i)$ leaves. Then*

1. *q_{i+1} and q_n are vertices of the convex hull C_{i+1} of the points $q_{i+1}, \dots, q_n$ ($i \geq 0$);*
2. *$q_i \notin C_{i+1}$ and the vertex q_{i+1} of C_{i+1} is visible from q_i;*
3. *The sequence $q_{i+1}, \mathrm{u}(q_{i+1}), \mathrm{u}^2(q_{i+1}), \dots$ is finite, its last term is q_n, and coincides with the sequence of vertices of C_{i+1} lying between q_{i+1} and q_n for the counterclockwise orientation of the boundary of C_{i+1}; in particular $\{(q_k, \mathrm{u}(q_k))\}_{k \neq n}$ is a plane tree whose root is q_n (dual version of the upper tree of EG)*
4. *$\mathrm{u}(q_i)$ is the first element z of the sequence $z' = q_{i+1}, \mathrm{u}(z'), \mathrm{u}^2(z'), \dots, q_n$ such that $q_i, z, \mathrm{u}(z)$ is counterclockwise.*
5. *A similar result holds for the operator l.* □

Horizon Trees

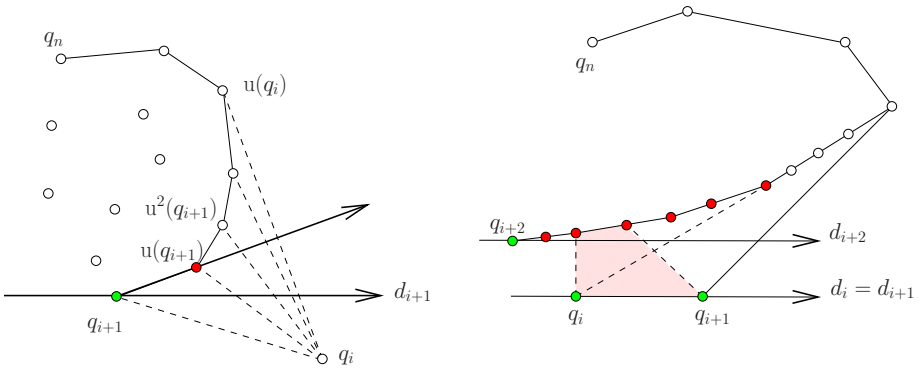
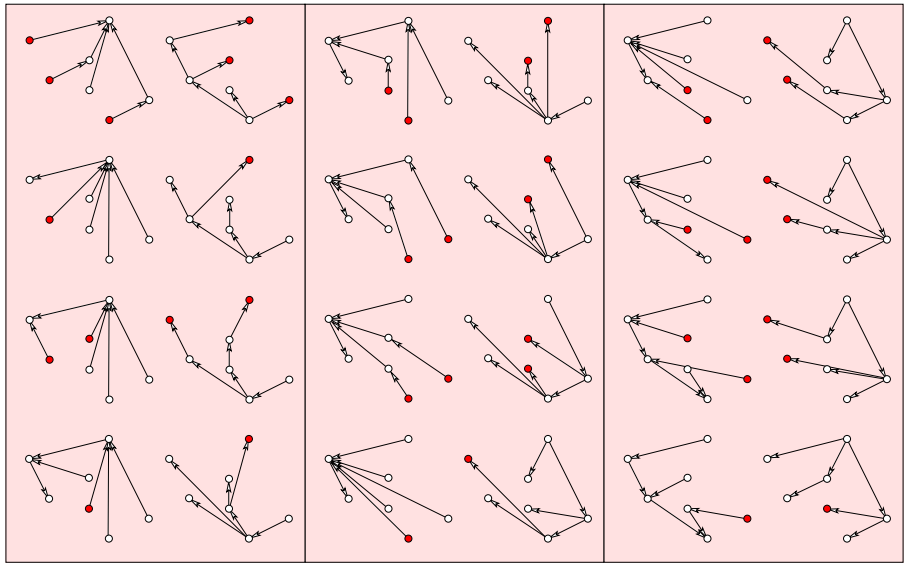
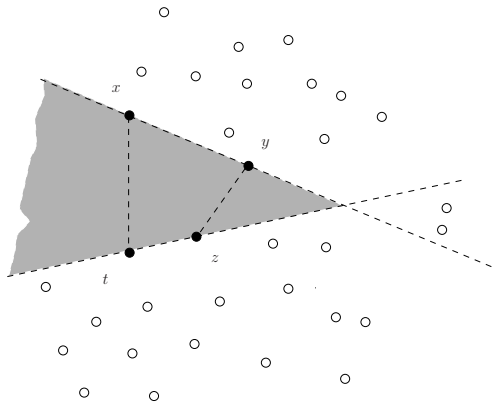


Figure 4: Blabla.

Horizon Trees

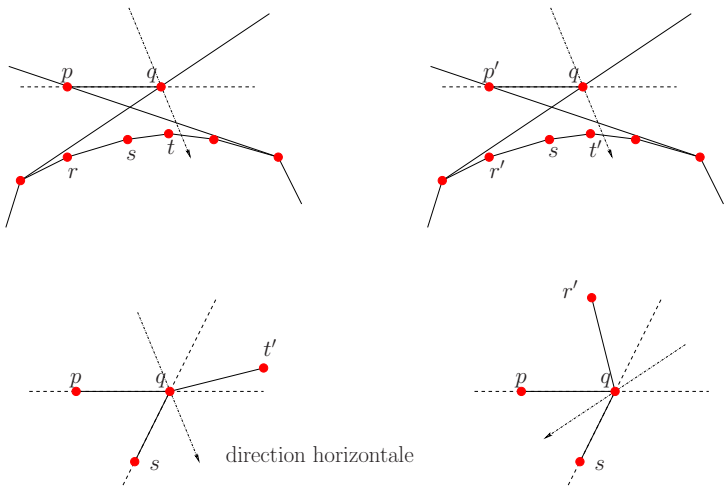


Complexity Analysis



Theorem 3. *The number of 4-uplets $xyzt$ of points of a set P of n points in general position such that (1-) x, y, z, t are in convex position and appear in this order when traversing clockwise their convex hull, and (2-) no point of P lies in the intersection of the right half-plane of the line going from x to y and the right half-plane of the line going from z to t, is a $O(n^2)$.* □

Proof



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Cell complex terminology

A finite regular cell complex is a topological space X and a partition of X into a finite number of 'cells' $\sigma_i \subset X$ (that is, the σ_i's are nonempty, pairwise-disjoint and $X = \bigcup \sigma_i$) such that

1. $(\overline{\sigma}_i, \overline{\sigma}_i \setminus \sigma_i) \sim (\mathbb{B}^n, \mathbb{S}^{n-1})$, for some $n = n(i)$,
2. each $\overline{\sigma}_i \setminus \sigma_i$ is a union of σ_j's.

Here $\overline{\sigma}_i$ denotes the closure of σ_i and $\sim$ denotes homeomorphism, $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid \langle x, x\rangle \leq 1\}$ is the unit ball of $\mathbb{R}^n$, and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \langle x, x\rangle = 1\}$ is the unit sphere of $\mathbb{R}^n$ ($\langle x, y\rangle$ is the scalar product $x_1y_1 + \cdots + x_ny_n$ of the points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of $\mathbb{R}^n$).

The union X^n of the σ_i's with $n(i) \leq n$ is called the n-skeleton.

The connected components of $X^n \setminus X^{n-1}$ are called n-cells (0-cells and 1-cells are called vertices and edges).

The dimension of the complex is the integer n such that $X = X^n$.

Finally we use the term *graph* as synonymous with 1-dimensionnal cell complex.

Poset terminology

A *partially ordered set* P (or *poset*, for short) is a set, together with a partial order relation denoted by $\preceq$. A *subposet* of P is a subset of P with the induced order. A special type of subposet is the *interval* $[x, y] = \{z \in P \mid x \preceq z \preceq y\}$. A poset P is called a *locally finite* poset if every interval of P is finite. If $x, y \in P$, then we say that y *covers* x if $x \prec y$ (i.e., $x \preceq y$ and $x \neq y$) and if no element $z \in P$ satisfies $x \prec z \prec y$. The *Hasse diagram* of a poset is the graph whose vertices are the elements of P and whose edges are the cover relations.

A *chain* is a poset in which any two elements are comparable. A subset C of a poset P is called a *chain* if C is a chain when regarded as a subposet of P. The chain C of P is *saturated* (or *unrefinable*) if there does not exist $z \in P \setminus C$ such that $x \prec z \prec y$ for some $x, y \in C$ and such that $C \cup \{z\}$ is a chain. An *antichain* is a subset A of a poset P such that any two distinct elements of A are incomparable. A *filter* is a subset I of P such that if $x \in I$ and $x \preceq y$, then $y \in I$. A *proper* filter of a poset $(X, \preceq)$ is a filter which is a nonempty proper subset of X.