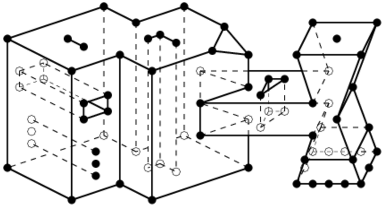
The 3D meshing problem

Input :

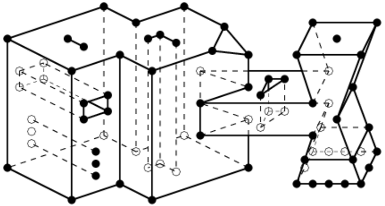
- a PLC piecewise linear complex C
- a bounded domain Ω to be meshed.
  Ω is bounded by facets in C

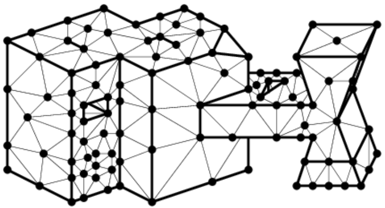


Output : a mesh of domain Ω
i. e. a 3D triangulation T such that

- vertices of C are vertices of T
- edges and facets C are union of faces in T
- the tetrahedra of T that are $\subset \Omega$ have controlled size and quality

Edges and facets of the input PLC are split into subconstraints which are edges and facets of the mesh, called constrained edges and facets.





3D Delaunay refinement

Use a 3D Delaunay triangulation
(in fact a 3D constrained Delaunay triangulation)

Constraints

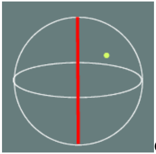
constrained edges are refined into Gabriel edges
encroached edges = edges which are not Gabriel edges

constrained facets are refined into Gabriel facets
encroached facets = facets which are not Gabriel facets

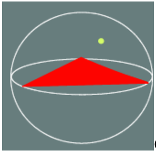
Tetrahedra

Bad tetrahedra are refined by circumcenter insertion.

Bad tetrahedra : radius-edge ratio
$\rho = \frac{\text{circumradius}}{l_{min}} \geq B$



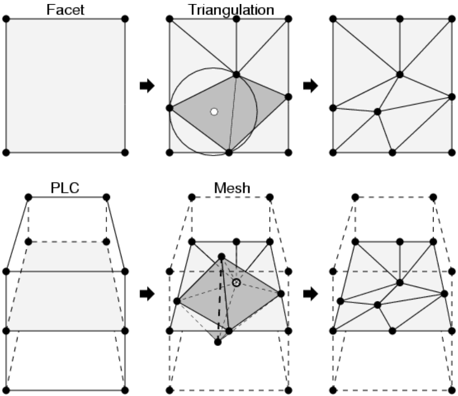
encroached edge



encroached facet

constrained facets

once contrained edges are refined into Gabriel edges
constrained facets are known : they are 2D Delaunay facets
A 2D Delaunay triangulation is maintained for each PLC facet



3D Delaunay refinement algorithm

- Initialization Delaunay triangulation of PLC vertices
- Refinement

Apply one the following rules, until no one applies.
Rule i has priority over rule j if $i < j$.

1. if there is an encroached constrained edge e, refine-edge(e)
2. if there is an encroached constrained facet f,
   conditionally-refine-facet(f) i.e.:
   c = circumcenter(f)
   if c encroaches a constrained edge e, refine-edge(e).
   else insert(c)
3. if there is a bad tetrahedra t,
   conditionally-refine-tet(t) i.e.:
   c = circumcenter(t)
   if c encroaches a constrained edge e, refine-edge(e).
   else if c encroaches a constrained facet f,
   conditionally-refine-facet(f).
   else insert(c)

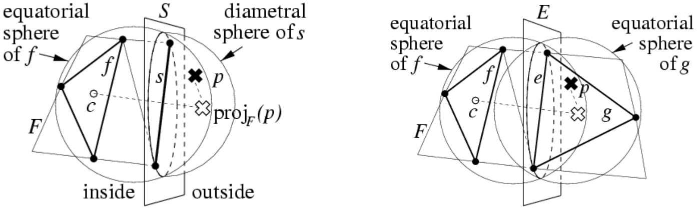
Refinement of constrained facets

Lemma (Projection lemma)

When a point p encroaches a constrained subfacet f of PLC facet F without constrained edges encroachment :

- *the projection p_F of p on the supporting hyperplan h_F of F, belongs to F*
- *p encroaches the mesh facet $g \subset F$ that contains p_F*

Proof.



□

The algorithm always refine a constrained facet including the projection of the encroaching point

3D Delaunay refinement theorem

Theorem (3D Delaunay refinement)

The 3D Delaunay refinement algorithm ends provided that ;

- *the upper bound on radius-edge ratio of tetrahedra is*

$$B > 2$$

- *all input PLC angles are* $> 90^o$
  dihedral angles : two facets of the PLC sharing an edge
  edge-facet angles : a facet and an edge sharing a vertex
  edge angles : two edges of the PLC sharing a vertex

Proof.

As in 2D, use a volume argument
to bound the number of Steiner vertices □

Proof of 3D Delaunay refinement theorem

Lemma (Lemma 1)

Any added (Steiner) vertex is inside or on the boundary of the domain Ω to be meshed

Proof.

as in 2D, because Steiner vertices are added when there is no encroached edge and no encroached facet. □

Proof of 3D Delaunay refinement theorem

Local feature size $\mathrm{lfs}(p)$

radius of the smallest disk centered in p and intersecting two disjoint elements of C.

Insertion radius r_v

length of the smallest edge incident to v, right after insertion of v, if v is inserted.

Parent vertex p of vertex v

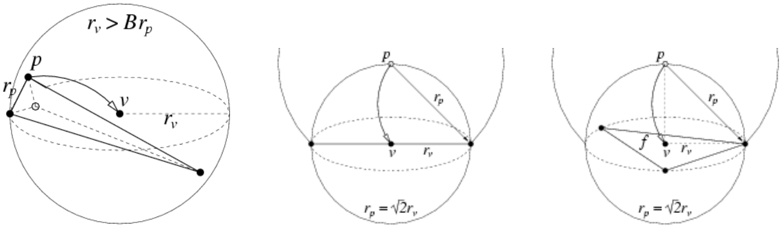
- if v is the circumcenter of a tet t
  p is the last inserted vertex of the smallest edge of t
- if v is inserted on a constrained facet or edge
  p is the encroaching vertex closest to v
  (p may be a mesh vertex or a rejected vertex)

Proof of 3D Delaunay refinement theorem

Lemma (Insertion radius lemma)

Let v be vertex of the mesh, with parent p, $r_v \geq \mathrm{lfs}(v)$ or $r_v \geq Cr_p$, with :

- *$C = B$ if v is a tetrahedra circumcenter*
- *$c = 1/\sqrt{2}$ if v is on a PLC edge or facet and p is rejected*



Refinement of constrained facets

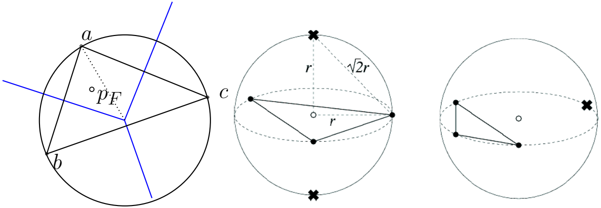
Refining the facet in F including the projection p_F of the encroaching point guarantees : $r_v \geq \frac{r_p}{\sqrt{2}}$

p encroaching point of a facet $f \subset F$

r_p insertion radius of p,

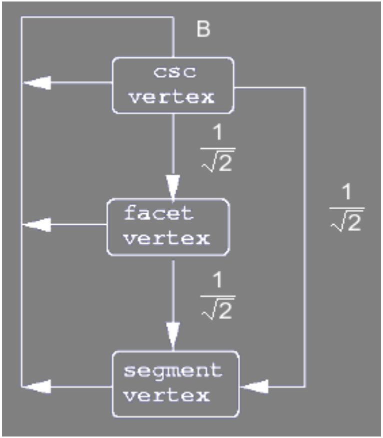
r_v insertion radius of the point v, $r_v = r$

$r_p \leq pa$ if $pa = \min\{pa, pb, pc\}$ $\quad \|pa\|^2 = \|pp_F\|^2 + \|p_F a\|^2 \leq 2r^2$



Proof of 3D Delaunay refinement theorem

Flow diagram of vertices insertion



Proof of 3D Delaunay refinement theorem

weighted density $d(v) = \frac{\mathrm{lfs}(v)}{r_v}$

Lemma (Weighted density lemma 1)

For any vertex v with parent p, if $r_v \geq C r_p$, $d(v) \leq 1 + \frac{d(p)}{C}$

Lemma (Weighted density lemma 2)

There are constants $D_e \geq D_f \geq D_t \geq 1$ *such that :*
for any tet circumcenter v, inserted or rejected, $d(v) \leq D_t$
for any facet circumcenter v, inserted or rejected, $d(v) \leq D_f$.
for any vertex v inserted in a PLSG edge, $d(v) \leq D_e$.
Thus, for any vertex of the mesh $r_v \geq \frac{\mathrm{lfs}(v)}{D_e}$

Proof of weighted density lemma

Assume wd lemma is true up to the insertion of vertex v,
p parent of v

- v is a tet circumcenter
  $r_v \geq B r_p \Longrightarrow d(v) \leq 1 + \frac{d_p}{B}$ assume $1 + \frac{D_e}{B} \leq D_t$ (1)
- v is on a PLC facette f
  - p is a PLC vertex or $p \in$ PLC face s' st $f \cap s' = \emptyset$
    $r_v = \mathrm{lfs}(v) \Longrightarrow d(v) \leq 1$
  - p is a tet circumcenter
    $r_v \geq \frac{r_p}{\sqrt{2}} \Longrightarrow d(v) \leq 1 + \sqrt{2} d_p$ assume $1 + \sqrt{2} D_t \leq D_f$ (2)
- v is on a PLC edge e
  - p is a PLC vertex or $p \in$ PLC face s' st $e \cap s' = \emptyset$
    $r_v = \mathrm{lfs}(v) \Longrightarrow d(v) \leq 1$
  - p is a tet or a facet circumcenter
    $r_v \geq \frac{r_p}{\sqrt{2}} \Longrightarrow d(v) \leq 1 + \sqrt{2} d_p$ assume $1 + \sqrt{2} D_f \leq D_e$ (3)

3D Delaunay refinement theorem

Proof of weighted density lemma (end)

There are $D_e \geq D_f \geq D_t \geq 1$ such that :

$1 + \frac{D_e}{B} \leq D_t \quad (1)$

$1 + \sqrt{2}D_t \leq D_f \quad (2)$

$1 + \sqrt{2}D_f \leq D_e \quad (3)$

$$\begin{aligned} D_e &= \left(3+\sqrt{2}\right)\frac{B}{B-2} \\ D_f &= \frac{(1+\sqrt{2}B)+\sqrt{2}}{B-2} \\ D_t &= \frac{B+1+\sqrt{2}}{B-2} \end{aligned}$$

Proof of 3D Delaunay refinement theorem (end)

Theorem (Relative bound on edge length)

Any edge of the mesh, incident to vertex v, has length l st :

$$l \geq \frac{\mathrm{lfs}(v)}{D_e + 1}$$

Proof.

as in 2D □

End of 3D Delaunay refinement theorem proof.

Using the above result on edge lengths,
prove an upper bound on the number of mesh vertices
as in 2D. □

Delaunay refinement

meshing domain with small angles

Algorithm Terminator 3D : Delaunay refinement + additionnal rules

1. Clusters of edges : refine edges in clusters along concentric spheres
2. when a facet f in PLC facet F is encroached by p
   and $\mathrm{circumcenter}(f)$ encroaches no constrained edge
   refine f iff
   - p is a PLC vertex or belongs to a PLC face s' st $f \cap s' = \emptyset$
   - $r_v > r_g$, where g is the most recently inserted ancestor of v.
3. when a constrained edge e is encroached by p
   e is refined iff
   - p is a mesh vertex
   - $\min_{w \in W} r_w > r_g$ where
     g is the most recently inserted ancestor of v
     W is the set of vertices that will be inserted if v is inserted.

Delaunay refinement

About terminator 3D

Remarks Notice that some constrained facets remain encroached

- using a constrained Delaunay triangulation is required to respect constrained facets.
  Fortunately, this constrained Delaunay triangulation exists because constrained edges are Gabriel edges.
- the final mesh may be different from the Delaunay triangulation of its vertices

Nearly degenerated triangles

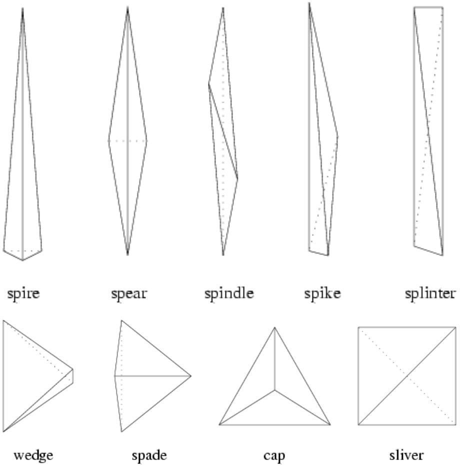
Radius-edge ratio $\rho = \frac{\text{circumradius}}{\text{shortest edge lentgh}}$
In both cases the radius-edge ratio is large

Nearly degenerated tetrahedra

Thin tetrahedra

Flat tetrahedra

Slivers : the only case in which radius-edge ratio ρ is not large



Slivers

Definition (Slivers)

A tetrahedra is a sliver iff
the radius-edge ratio is not too big $\rho = \frac{r}{l} \leq \rho_0$
yet, the volume is too small $\sigma = \frac{V}{l^3} \leq \sigma_0$
r = circumradius, l = shortest edge length, V = volume

Remark

Tetrahedra with bounded radius-edge ratio, that are not slivers have a bounded radius-radius ratio :
$\rho \leq \rho_0 \text{ and } \sigma > \sigma_0, \Longrightarrow \frac{\mathrm{r_{circ}}}{\mathrm{r_{insc}}} \leq \frac{\sqrt{3}\rho_0^3}{\sigma_0}$

Proof.

area of facets of t : $S_i \leq \frac{3\sqrt{3}}{4} r_{circ}^2$

$$\sqrt{3} r_{circ}^2 r_{insc} \geq \sum_{i=1}^4 \frac{1}{3} S_i r_{insc} = V \geq \sigma_0 l^3 \geq \sigma_0 \left(\frac{r_{circ}}{\rho_0}\right)^3$$

□

Delaunay meshes with bounded radius-edge ratio

Theorem (Delaunay meshes with bounded radius-edge ratio)

Any Delaunay mesh with bounded radius-edge ratio is such that :

1. *The ratio between the length of the longest edge and the length of shortest edge incident to a vertex v is bounded.*
2. *The number of edges, facets or tetrahedra incident to a given vertex is bounded*

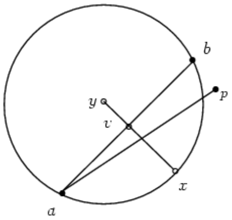
Delaunay meshes with bounded radius-edge ratio

Lemma

In a Delaunay mesh with bounded radius-edge ratio ($\rho \leq \rho_0$), edges ab, ap incident to the same vertex and forming an angle less than $\eta_0 = \arctan\left[2\left(\rho_0 - \sqrt{\rho_0^2 - 1/4}\right)\right]$ *are such that* $\frac{\|ab\|}{2} \leq \|ap\| \leq 2\|ab\|$

Proof.

$\Sigma(y, r_y) =$ Intersection of the hyperplan spanned by (ap, ab) with the circumsphere of a tetrahedron incident to ab



$$\|xv\| = r_y - \sqrt{r_y^2 - \|ab\|^2/4}$$

$$\|xv\| \geq \left(\rho_0 - \sqrt{\rho_0^2 - 1/4}\right)\|ab\|$$

$$(\widehat{ab, ax}) = \arctan\left(\frac{2\|xv\|}{\|ab\|}\right) \geq \eta_0$$

$$(\widehat{ab, ap}) \leq \eta_0 \implies \|ap\| \geq \|ax\| \geq \frac{\|ab\|}{2}$$

□

Delaunay meshes with bounded radius-edge ratio theorem

proof of Part 1

ρ_0 radius-edge ratio bound

$m_0 = \frac{2}{(1-\cos(\eta_0/4))}$

$\eta_0 = \arctan\left[2\left(\rho_0 - \sqrt{\rho_0^2 - 1/4}\right)\right]$

$\nu_0 = 2^{2m_0-1}\rho_0^{m_0-1}$

Two mesh edges ab and ap incident to a are such that :
$\frac{\|ab\|}{\nu_0} \leq \|ap\| \leq \nu_0\|ab\|$

Proof.

$\Sigma(a,1)$ unit sphere around a
Max packing on Σ of spherical caps with angle $\eta_0/4$
There is at most m_0 spherical caps
Doubling the cap's angles form a covering of Σ.
Graph G = traces on $\Sigma(a,1)$ of edges and facets incident to a.
Path in G from ab to ap, ignore detours when revisiting a cap.
The path visits at most m_0 and crosses at most $m_0 - 1$ boundary. □

The number of edges incident to a given vertex is bounded by $\delta_0 = (2\nu_0^2 + 1)^3$

Proof.

ap : shortest edge incident to a, let $\|ap\| = 1$
ab : longest edge incident to a, $\|ap\| \leq \nu_0$
for any vertex c adjacent to a, $1 \leq \|ac\| \leq \nu_0$
for any vertex d adjacent to c, $\|cd\| \geq \frac{1}{\nu_0}$
Spheres $\Sigma_c(c, \frac{1}{2\nu_0})$ are empty of vertices except c, disjoint and included in $\Sigma(a, \nu_0 + \frac{1}{2\nu_0}))$

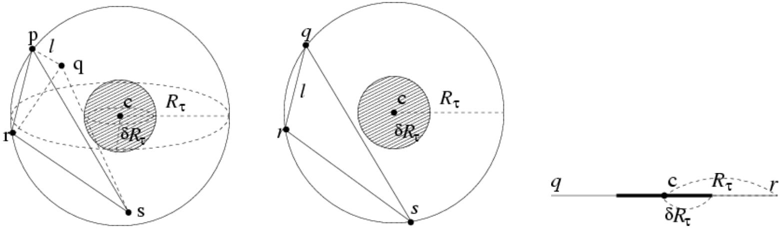
$$V_\Sigma = \frac{4}{3}\pi \left(\nu_0 + \frac{1}{2\nu_0}\right)^3 = \left(2\nu_0^2 + 1\right)^3 V_{\Sigma_c}$$

□

Sliver elimination

Method of Li [2000]
Choose each Steiner vertices in a refinement region :
Refinement region

refining a tetrahedra t with circumsphere (c_t, r_t) :	3D ball $(c_t, \delta r_t)$
refining a facet f with circumcircle (c_f, r_f) :	2D ball $(c_f, \delta r_f)$
refining an edge (c_s, r_s)	1D ball $(c_s, \delta r_s)$



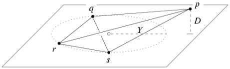
Sliver lemma

Definition (Slivers)

r = circumradius, l = shortest edge length, V = volume
$\rho = \frac{r}{l} \leq \rho_0 \quad \sigma = \frac{V}{l^3} \leq \sigma_0$

Lemma

If pqrs is a tet with $\sigma \leq \sigma_0$, $\frac{d}{r_y} \leq 12\sigma_0$
d : distance from p to the hyperplan of qrs
r_y : circumradius of triangle qrs



Proof.

$$\sigma l^3 = V = \frac{1}{3} S d \geq \frac{1}{3}\left(\frac{1}{2} l^2 \frac{l}{2r_y}\right) d = \frac{l^3}{12 r_y} d$$

□

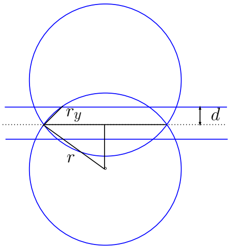
Sliver lemma

Lemma (Sliver lemma)

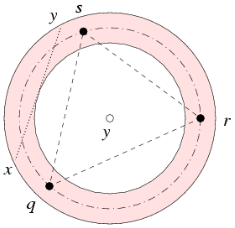
Let $\Sigma(y, r_y)$ be the circumcircle of triangle qrs.
If the tet pqrs is a sliver, $d(p, \Sigma(y, r_y)) \leq \gamma_2 r_y$
with $\gamma_2 = 48\sigma_0\rho_0$.

r circumradius of *pqrs*
H hyperplane of *pqr*

$$\begin{aligned} d(p,H) &\leq 12\sigma_0 r_y \\ r &\leq \sqrt{3}\rho_0 r_y \\ d(p,\Sigma(y,r_y)) &\leq \frac{d(p,H)}{\sin\theta} \\ \sin\theta &\approx \frac{r_y}{r} \\ d(p,\Sigma(y,r_y)) &\leq\approx 12\sqrt{3}\sigma_0\rho_0 \end{aligned}$$



Sliver elimination



Forbidden torus

For any triangle qrs,
p should not be in a torus
of volume $V(\mathrm{torus}(qrs))$:
$V(\mathrm{torus}(qrs)) \le \gamma_3 r_y^3$
$\gamma_3 = 2\pi^2 (48\sigma_0\rho_0)^2$

Forbidden area on any plane h

$S(\mathrm{torus}(qrs) \cap h) \le \gamma_4 r_y^2 \qquad \gamma_4 = 192\,(\pi\sigma_0\rho_0)$

$S(\mathrm{torus}(qrs) \cap h) \le \pi(r_y + d)^2 - \pi(r_y - d)^2 = 4\pi d r_y$
$d = d(p, \Sigma(y, r_y)) \le 48\sigma_0\rho_0 r_y$

Forbidden length on any line l

$L(\mathrm{torus}(qrs) \cap l) \le \gamma_5 r_y \qquad \gamma_5 = 16\sqrt{3\sigma_0\rho_0}$

$L(\mathrm{torus}(qrs) \cap h) \le 2\sqrt{(r_y + d)^2 - (r_y - d)^2} = 4\sqrt{r_y d}$

Sliver elimination

Main Idea

Start from a Delaunay mesh with bounded edge-radius ratio
Then refine bad tets ($\rho > \rho_0$) and slivers ($\rho \leq \rho_0, \sigma \leq \sigma_0$)
choosing refinement point in the refinement regions
avoiding forbidden volumes, areas and segments

When refining a mesh element τ (τ my be a tet, a facet or an edge)
it is not always possible to avoid producing new slivers
but it is possible to avoid producing small slivers,
i. e. slivers *pqrs* with circumradius $\mathrm{circumradius}(pqrs) \leq Cr_\tau$
where r_τ is the smallest circumradius of τ.

Lemma

For any refinement region $(c_\tau, \delta r_\tau)$
there is a finite number of facets (qrs)
such that, for a point $p \in (c_\tau, \delta r_\tau)$
tet pqrs is a sliver with $\mathrm{circumradius}(pqrs) \leq Cr_\tau$

Lemma

For any refinement region $(c_\tau, \delta r_\tau)$
there is a finite number of facets (qrs)
such that, for a point $p \in (c_\tau, \delta r_\tau)$
tet $pqrs$ *is a sliver with* $\mathrm{circumcircle}(pqrs) \leq Cr_\tau$

Proof.

$$\mathrm{circumradius}(pqrs) \leq Cr_\tau \Longrightarrow \|pq\|, \|pr\|, \|ps\| < 2Cr_\tau$$
$$q, r, s \in \text{ball } \Sigma\,(c_\tau, r_1)\,, \quad r_1 = (2C+\delta)r_\tau \qquad (1)$$

$$\|pq\|, \|pr\|, \|ps\| \geq (1-\delta)r_\tau \Longrightarrow \mathrm{circumradius}(pqrs) \geq \frac{(1-\delta)r_\tau}{2}$$
$$\rho(pqrs) \leq \rho_0 \Longrightarrow \|qr\|, \|rs\|, \|sq\| \geq \frac{\mathrm{circumradius}(pqrs)}{\rho(pqrs)} \geq \frac{(1-\delta)r_\tau}{2\rho_0}$$

When a sliver is refined, radius-edge ratios are bounded by ρ_0
hence, any edge incident to q has length $l > \frac{(1-\delta)r_\tau}{2\rho_0\nu_0} = 2r_2$ (2)

number W of slivers to avoid when picking p in $(c_\tau, \delta r_\tau)$

$$(1) + (2) \Longrightarrow W = \left(\frac{r_1+r_2}{r_2}\right)^3 = \left(\frac{(2C+\delta)4\rho_0\nu_0+(1-\delta)}{(1-\delta)}\right)^3$$

□

Sliver elimination

- Initial phase

Build a bounded radius-edge ratio mesh
using usual Delaunay refinement

- Sliver elimination phase

Apply one of the following rules, until no one applies
Rule i has priority over rule j if $i < j$.

1. if there is an encroached constrained edge e,
   sliver-free-refine-edge(e)
2. if there is an encroached constrained facet f,
   sliver-free-conditionally-refine-facet(f)
3. if there is a tet t with $\rho \geq \rho_0$,
   sliver-free-conditionally-refine-tet(t)
4. if there is a sliver t,
   sliver-free-conditionally-refine-tet(t)

Sliver elimination

Sliver-free versions of refine functions

sliver-free-refine-edge(e)

sliver-free-conditionally-refine-facet(f)

sliver-free-conditionally-refine-tet(t)

- pick q sliver free in refinement region
- if q encroaches a constrained edge e,
  sliver-free-refine-edge(e).
- else if q encroaches a constrained facet f,
  sliver-free-conditionally-refine-facet(f).
- else insert(q)

picking q sliver free in refinement region means :

- pick a random point q in refinement region
- while q form small slivers
  pick another random point q in refinement region

Sliver elimination

Theorem

If the hypothesis of Delaunay refinement theorem are satisfied and if the constants δ, ρ_0 and C are such that

$$\frac{(1-\delta)^3\rho_0}{2} \geq 1 \text{ and } \frac{(1-\delta)^3 \mathrm{C}}{4} \geq 1$$

the sliver elimination phase terminates
yielding a sliver free bounded radius-edge ratio mesh
i.e. for any tetrahedron $\rho \leq \rho_0$ and $\sigma \geq \sigma_0$

Proof.

Two lemmas to show that
if l_1 is the shortest edge length before sliver elimination phase
the shortest edge length after sliver elimination phase is $l_2 = \frac{(1-\delta)^3 l_1}{4}$ □

Original mesh = bounded radius-edge ratio mesh obtained in first phase
original sliver = sliver of the original mesh

Lemma

Any point q whose insertion is triggered by an original sliver, has an insertion radius $r_q \geq l_2$ with $l_2 = \frac{(1-\delta)^3 l_1}{4}$

Proof.

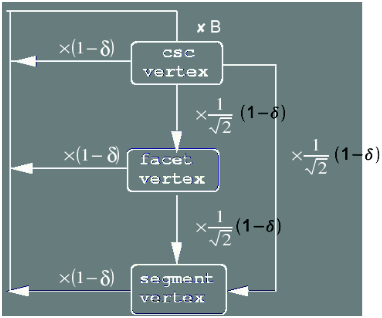
Assume an original sliver t with circumradius r_t
is eliminated by inserting a point q in a refinement region $(v, \delta r_v)$
of either an original sliver, or a constrained facet, or a constrained edge.

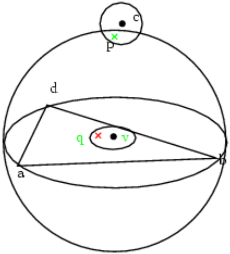
$$l_2 \geq (1-\delta) r_v \geq \left| \begin{array}{l} (1-\delta) r_t \\ (1-\delta)^2 \frac{r_t}{\sqrt{2}} \\ (1-\delta)^3 \frac{r_t}{2} \end{array} \right. \qquad r_t \geq \frac{l_1}{2}$$

□

Insertion radius

Flow diagram





$$r_q \geq (1-\delta) r_v$$
$$r_p \geq (1-\delta) r_c$$
$$r_p \leq \min(\|pa\|, \|pb\|, \|pd\|) \leq \sqrt{2} r_v$$
$$r_q \geq \frac{(1-\delta)^2}{2} r_c$$

Lemma

If $\frac{(1-\delta)^3\rho_0}{2} \geq 1$ *and* $\frac{(1-\delta)^3 C}{4} \geq 1$
any vertex inserted during the sliver elimination phase has an insertion radius at least $l_2 = \frac{(1-\delta)^3 l_1}{4}$.

Proof.

By induction, let t be the tetrahedron that triggers the insertion of p
- done if t is an original sliver
- otherwise

$$l_2 \geq (1-\delta) r_v \geq \left| \begin{array}{l} (1-\delta) r_t \\ (1-\delta)^2 \frac{r_t}{\sqrt{2}} \\ (1-\delta)^3 \frac{r_t}{2} \end{array} \right. \quad \text{with} \quad \begin{array}{l} r_t \geq \rho_0 l_1 \\ r_t \geq C r'_t \geq C \frac{l_1}{2} \end{array}$$

□

Sliver elimination

condition for termination :

choose δ and C such that

$\frac{(1-\delta)^3\rho_0}{2} \geq 1$

$\frac{(1-\delta)^3 C}{4} \geq 1$

condition for possibility of sliver-free picking:

choose σ_0 such that :

$W\gamma_3(Cr_t)^3 \leq \frac{4}{3}\pi(\delta r_t)^3 \quad \gamma_3 = 2\pi^2 \left(48\sigma_0\rho_0\right)^2$

$W\gamma_4(Cr_t)^2 \leq \pi(\delta r_t)^2 \quad \gamma_4 = 192 \left(\pi\sigma_0\rho_0\right)$

$W\gamma_5(Cr_t) \leq (\delta r_t) \quad \gamma_5 = 16\sqrt{3\sigma_0\rho_0}$

where $W = f(C, \rho_0, \delta)$ is the number of slivers to avoid