Triangulations and Meshes

Outline

- Triangulations, Delaunay triangulations
  Voronoi dagrams, the space of spheres
  Regular triangulations and power diagrams

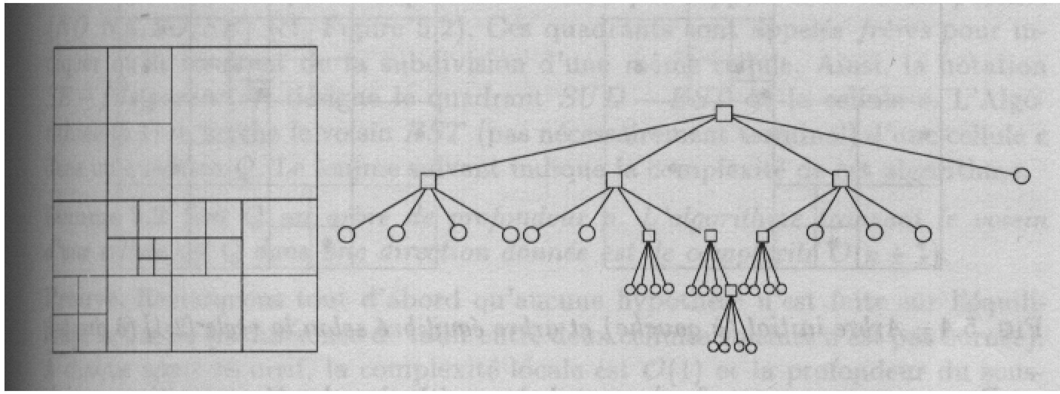
- Constrained and Delaunay constrained triangulations

- Meshing using Delaunay refinement

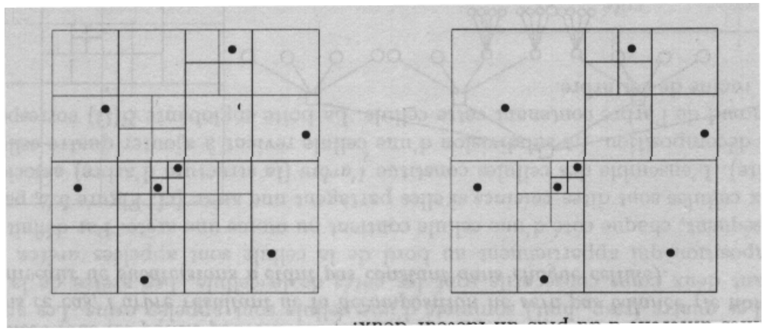
- Meshing using other methods (octree based, advancing front)

- Quality of meshes for linear interpolation
  and finite elements computation

Mesh generation : quadtree-octree based methods

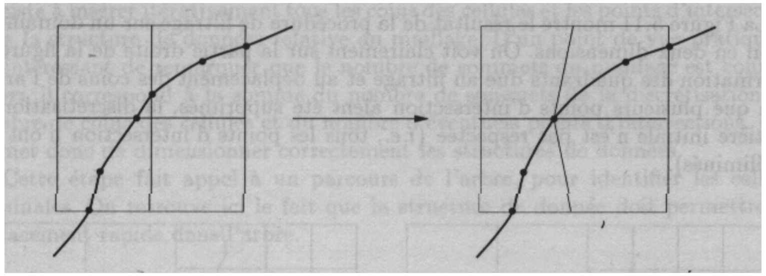


balanced tree



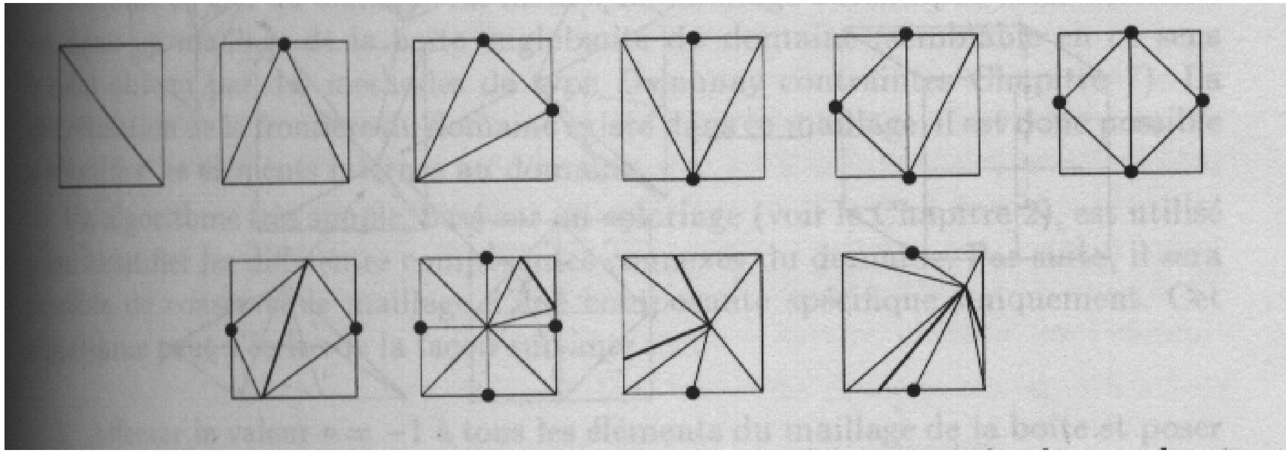
Mesh generation : quadtree-octree based methods

1. Build the octree from the bounding box
   by recursive subdivision
   until each terminal cell has a connected intersection with constraints
2. Balance the octree
3. Add vertices at the intersections
   between octree subdivision and constraints
4. Filter added vertices
5. Triangulate terminal cells
6. Optimize the mesh

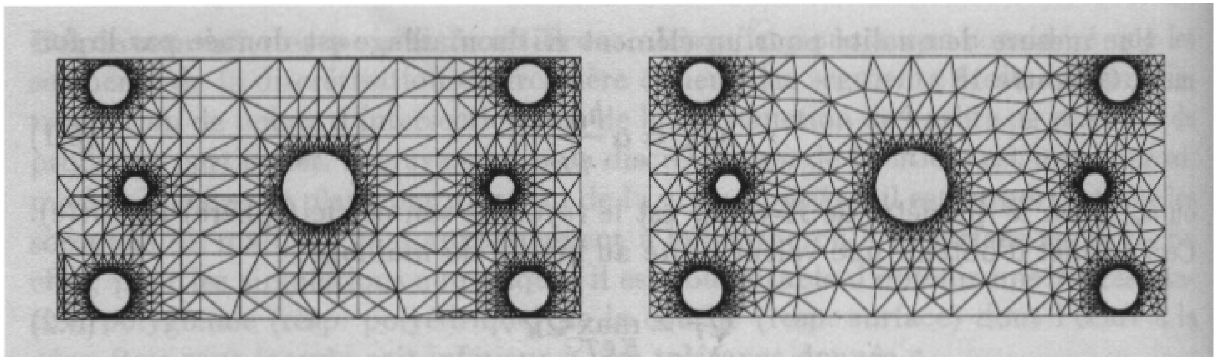


Mesh generation : quadtree-octree based methods

Triangulation of terminal cells



Mesh generation : quadtree-octree based methods



Advantages

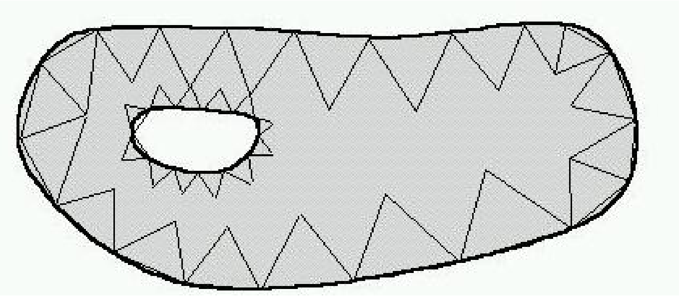
- octree methods can generate size-optimal meshes
  with guaranteed quality elements

Drawbacks

- Too many mesh elements in practice
- Octree directions remain visible in the final mesh
- Constraints and boundaries are subdivided
- Poor quality of mesh simplices adjacent to constrained elements

Mesh generation : advancing front methods

1. Mesh of the domain boundary
   = intial front
2. While front is not empty
   - choose a front facet F_i
   - compute an opposite vertex P_i
   - add simplex $conv(F_i, P_i)$ to the mesh
     and update the front
3. Optimize the mesh

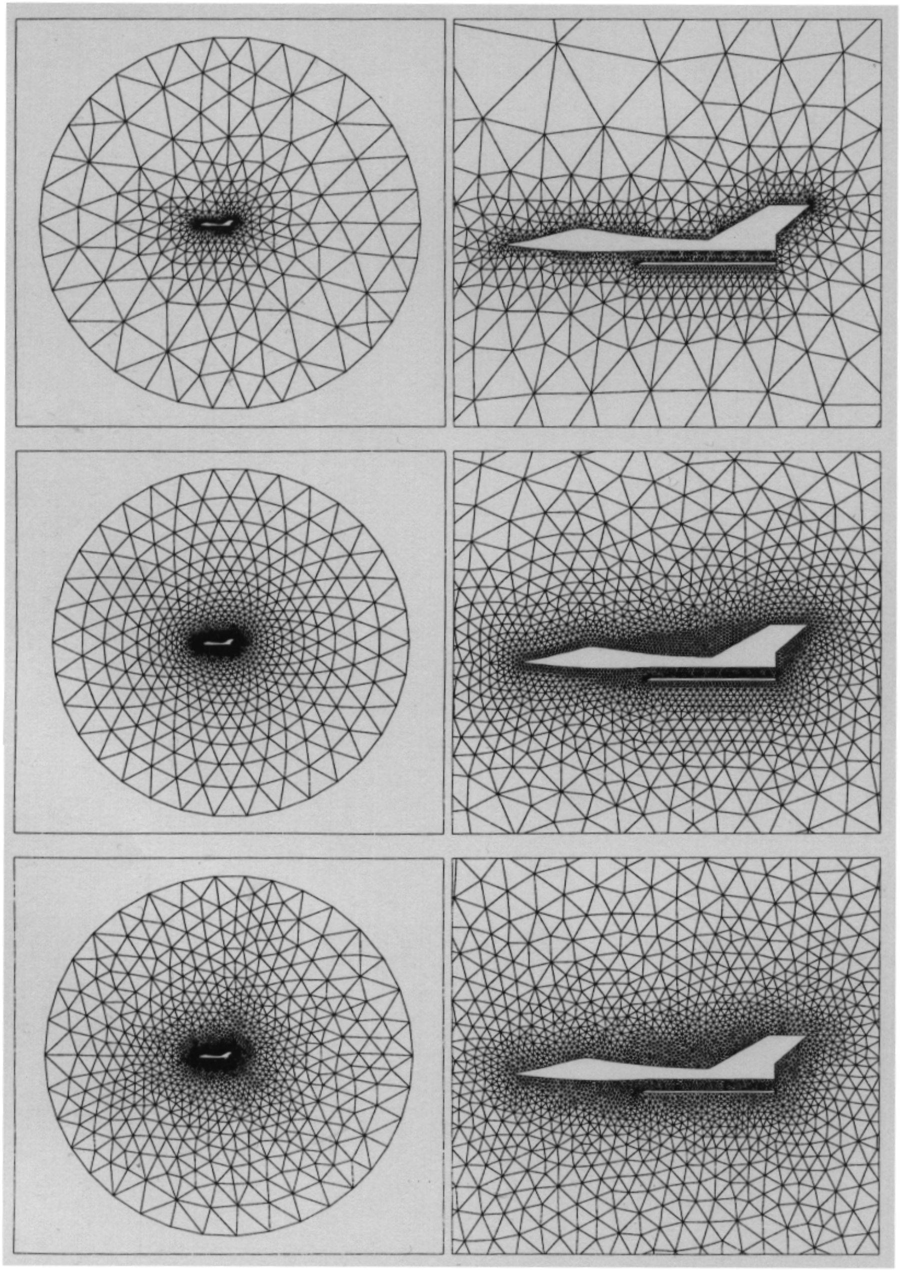


Mesh generation : advancing front methods

Computation of vertex P_i opposite to facet F_i
- $conv(F_i, P_i)$ has a good shape
- P_i is not too close to an existing vertex
  otherwise this vertex is choosen as P_i
- $conv(F_i, P_i)$ intersect no existing mesh facet

Advantages
- the initial boundary mesh is preserved
- good quality of mesh cells incident to constrained elements

Drawbacks
- complexity : intersection tests
- dead lock situations may be encountered
  no guarantee of termination



Quadtree

Advancing front

Delaunay refinement

Mesh generation : the unit march

Data : a boundary mesh + a sizing field field

sizing field is often interpolated from an auxiliary background mesh

Start from a coarse mesh.

Refinement loop :

1. Compute edge lengths

$$\text{isotropic metric} \quad l_{ab} = \; d_{ab} \int_0^1 \frac{dt}{h(t)}$$

$$\text{anisotropic metric} \quad l_{ab} = \; \int_0^1 \sqrt{\overline{ab}^T \,\overline{\overline{M(a+bt)}}\, \overline{ab}}\, dt$$

2. Compute candidate vertices to subdivide long edges.
3. Filter candidate vertices.
3. Insert remaining candidates in the mesh
   using constrained Delaunay algorithm.

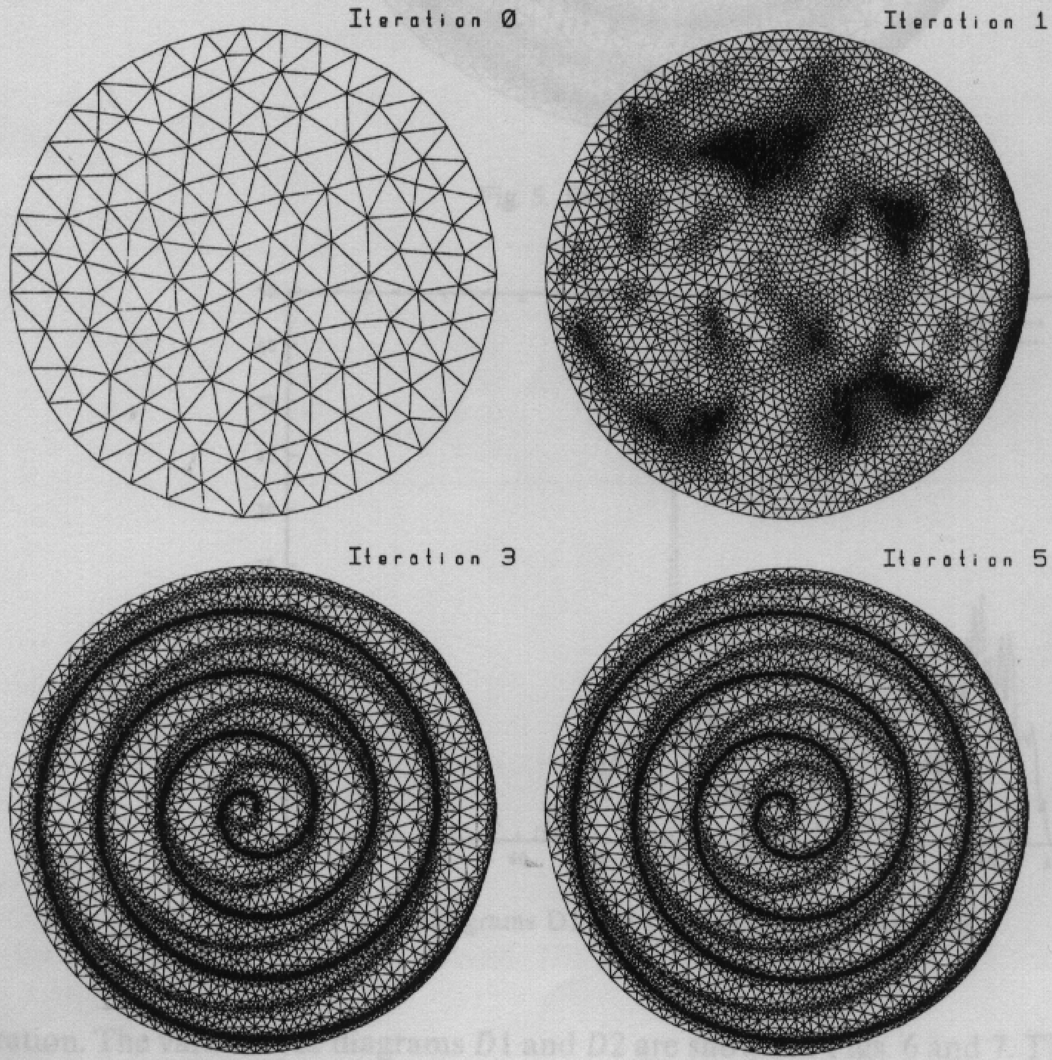
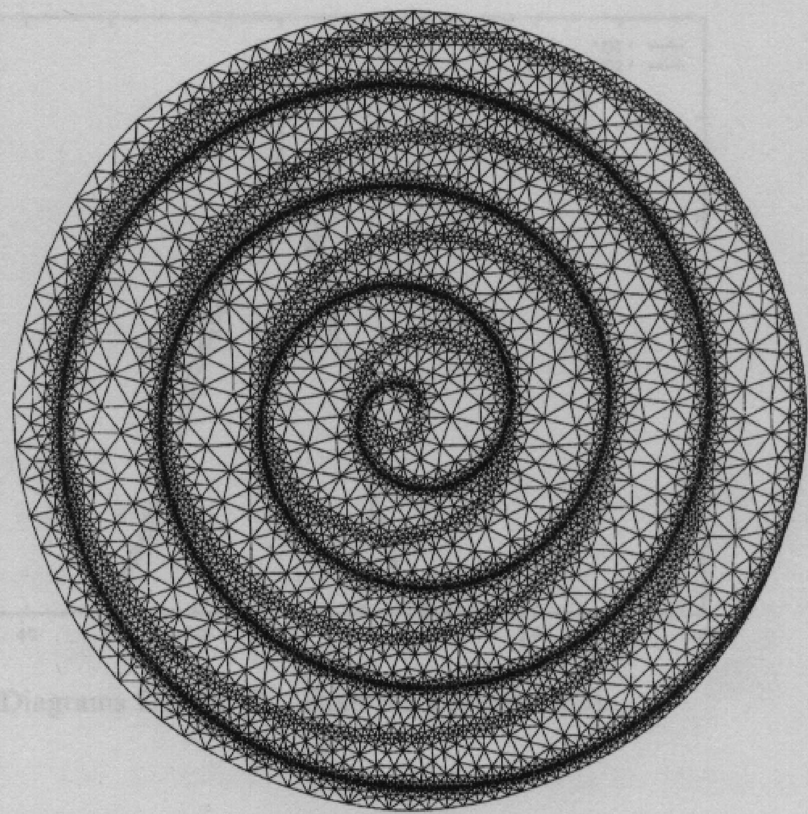


Fig. 4. Example 2 (isotropic case).



Mesh generation : adaptative meshes

1. Build the initial mesh T_i
2. Compute the solution h_i of the PDE using T_i
3. Estimate the local error δ_i on h_i
   STOP if error bound is met
4. Otherwise build a new mesh T_{i+1}
   using a sizing field yield by error estimation δ_i
5. go back to step 2 with $i = i + 1$.

Linear interpolation

T a (2D or 3D) mesh

$f(p)$ continuous scalar function defined on the domain $\Omega(T)$

$g(p)$ piecewise linear approximation of $f(p)$ such that :

$g(v) = f(v)$ for any vertex v of T

Interpolation error on cell $t \in T$

$\|f - g\|_\infty = \max_{p \in t} |f(p) - g(p)|$

$\|\nabla f - \nabla g\|_\infty = \max_{p \in t} \|\nabla f(p) - \nabla g(p)\|$

f is assumed to have a bounded curvature on t

$\forall \boldsymbol{d}$ with $\|d\| = 1, \;\; f''_{\boldsymbol{d}}(p) = \boldsymbol{d}^T H(p) \boldsymbol{d} \leq c_t$

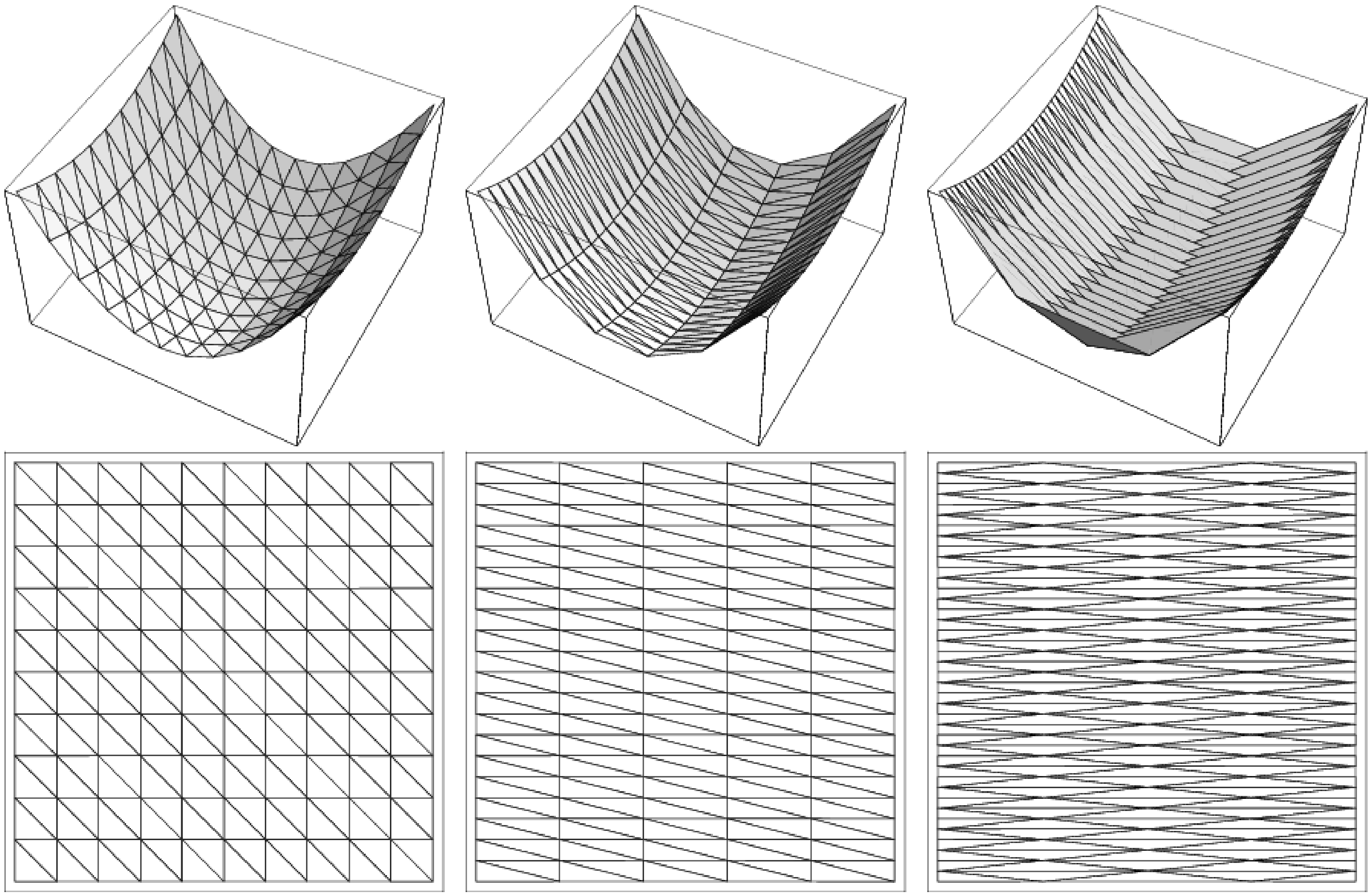
Bounds on 2D interpolation errors

	$\|f-g\|_\infty$	$\|\nabla f - \nabla g\|_\infty$
Upper bound	$c_t \frac{r_{\text{mc}}{}^2}{2}$	$c_t \frac{l_{\text{max}}\, l_{\text{med}}\, (l_{\text{min}}+4 r_{\text{in}})}{4A}$
Weak upper bound	$c_t \frac{l_{\text{max}}^2}{6}$	$c_t \frac{3 l_{\text{max}}\, l_{\text{med}}\, l_{\text{min}}}{4A}$
Lower bound	$c_t \frac{r_{\text{mc}}{}^2}{2}$	$c_t \max\left\{ r_{\text{circ}}, h_{\text{max}}, \sqrt{l_{\text{max}}^2 - h_{\text{med}}^2} \right\}$

r_{mc} radius of the smallest enclosing circle (min containment radius)

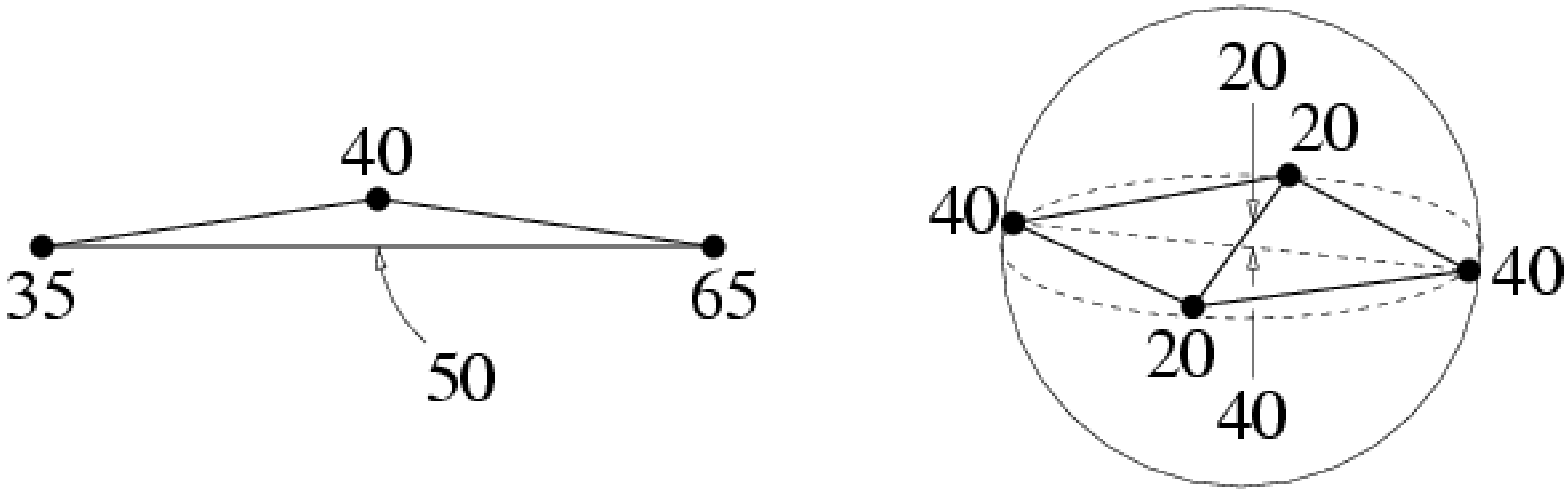
$A = \frac{1}{2} r_{\text{in}} (l_{\text{max}} + l_{\text{med}} + l_{\text{min}}) = \frac{1}{2} l_{\text{med}}\, l_{\text{min}} \sin\theta_{\text{max}} \Longrightarrow r_{\text{in}} \leq \frac{l_{\text{min}}}{2}$

$\frac{l_{\text{max}}\, l_{\text{med}}\, l_{\text{min}}}{4A} = \frac{l_{\text{max}}}{2\sin\theta_{\text{max}}} = r_{\text{circ}}$



Linear interpolation

Large angles are harmfull for the gradient error $\|\nabla f - \nabla g\|$



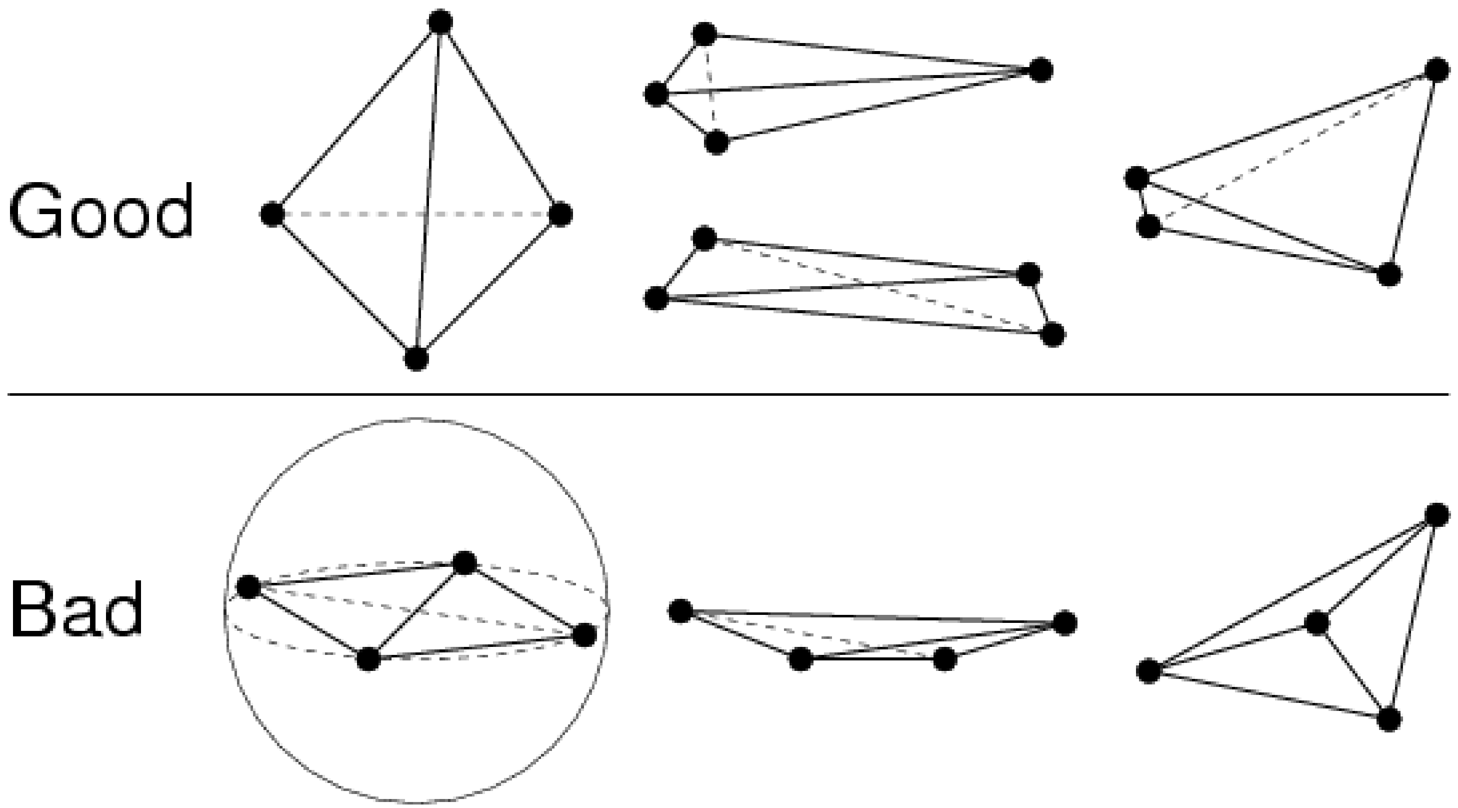
Bounds on 3D interpolation errors

	$\|f-g\|_\infty$	$\|\nabla f - \nabla g\|_\infty$
Upper bound	$c_t \frac{r_{\text{mc}}^2}{2}$	$c_t \frac{\frac{1}{6V}\sum_{1\le i\le j\le 4} A_i A_j l_{ij}^2 + \max_i \sum_{j\ne i} A_j l_{ij}}{\sum_{m=1}^4 A_m}$
Weak upper bound	$c_t \frac{3 l_{\text{max}}^2}{16}$	$c_t \frac{\sum_{1\le i\le j\le 4} A_i A_j l_{ij}^2}{2V \sum_{m=1}^4 A_m}$
Lower bound	$c_t \frac{r_{\text{mc}}^2}{2}$	$c_t r_{\text{circ}}$

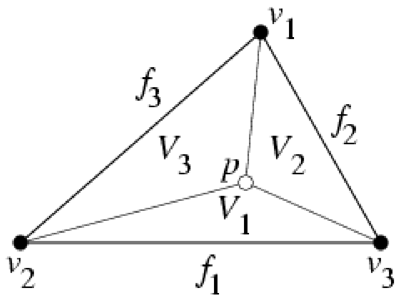
for a tetrahedra,

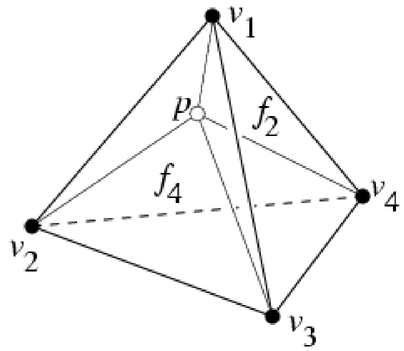
$$c_t \frac{\sum_{1\le i\le j\le 4} A_i A_j l_{ij}^2}{6V \sum_{m=1}^4 A_m} = c_t \frac{\sum_{1\le i\le j\le 4} l_{ij}^2 l_{kl} / \sin\theta_{kl}}{4 \sum_{m=1}^4 A_m}$$

Bounds on 3D interpolation errors



Barycentric coordinates





$v_1, v_2, \ldots, v_{d+1}$ vertices of a d-simplex t

Barycentric coordinates de p $\quad p = \sum_{i=1}^{d+1} \omega_i v_i, \quad \sum_{i=1}^{d+1} \omega_i = 1$

t_i simplex obtained when vertex v_i of t
is replaced by p

$V_i(p)$ volume of t_i,

V volume of t $\quad \omega_i(p) = \frac{V_i(p)}{V}$

Linear interpolation of f on t $\quad g(p) = \sum_{i=1}^{d+1} \omega_i(p) f(v_i)$

Gradient of barycentric coordinates

a_i altitude of t from v_i

$a_i(p)$ altitude of t_i from p

$$\begin{aligned}\omega_i(p) &= \frac{V_i(p)}{V} = \frac{a_i(p)}{a_i} \\ |\nabla\omega_i(p)| &= \frac{1}{a_i}|\nabla a_i(p)| = \frac{1}{a_i}\end{aligned}$$

$$\sum_{i=1}^{d+1}\omega_i = 1 \quad \Longrightarrow \quad \sum_{i=1}^{d+1}\nabla\omega_i = 0$$

$\forall$ vector $\boldsymbol{d}$,

$$\boldsymbol{d}\cdot p = \sum_{i=1}^{d+1}\omega_i(p)(v_i\cdot\boldsymbol{d}) \quad \Longrightarrow \quad \boldsymbol{d} = \nabla(\boldsymbol{d}\cdot p) = \sum_{i=1}^{d+1}(v_i\cdot\boldsymbol{d})\nabla\omega_i(p)$$

Gradient of barycentric coordinates

For a triangle t

$$
\begin{aligned}
\nabla\omega_i \cdot \nabla\omega_j &= \frac{1}{2}\Big(|\nabla\omega_i + \nabla\omega_j|^2 - |\nabla\omega_i|^2 - |\nabla\omega_j|^2\Big) \\
&= \frac{1}{2}\Big(|-\nabla\omega_k|^2 - |\nabla\omega_i|^2 - |\nabla\omega_j|^2\Big) \\
&= \frac{1}{2a_k^2} - \frac{1}{2a_i^2} - \frac{1}{2a_j^2} \\
&= \frac{l_k^2 - l_i^2 - l_j^2}{8A^2}
\end{aligned}
$$

Bounds on interpolation error

g linear interpolation of f on t

$e(p) = f(p) - g(p)$

e vanishes on vertices of t

e, f same curvature $\leq c_t$

$$e(q) = e(p) + \int_p^q \nabla e(u) \cdot du$$

$$e(q) = e(p) + \int_0^1 \nabla e(u(j)) \cdot (q-p)\, dj \qquad u(j) = (1-j)p + jq$$

$$= e(p) + \nabla e(p) \cdot (q-p) + \int_0^1 \int_0^j (q-p)^T H((u(k))\,(q-p)\, dk\, dj$$

$$= e(p) + \nabla e(p) \cdot (q-p) + \frac{1}{2}(q-p)^T \mathcal{H}\,(q-p)$$

with $\mathcal{H} = 2\int_0^1 \int_0^j H((u(k))\, dk\, dj$

and $\|(q-p)^T \mathcal{H}(q-p)\| \leq c_t \|q-p\|^2$

Bounds on interpolation error

$e(q) = e(p) + \nabla e(p) \cdot (q - p) + \frac{1}{2}(q-p)^T \mathcal{H}\,(q-p)$

At vertex $q = v_i$, the error vanishes $e(q) = e(v_i) = 0$

$$
\begin{aligned}
e(p) &= e_i(p) = -\nabla e(p) \cdot (v_i - p) - \frac{1}{2}(v_i - p)^T \mathcal{H}_i\,(v_i - p) \\
e(p) &= \sum_i \omega_i(p) e(p) = \sum_i \omega_i(p) e_i(p) \\
&= -\frac{1}{2} \sum_i \omega_i(p) (v_i - p)^T \mathcal{H}_i\,(v_i - p) \\
|e(p)| &\leq \frac{c_t}{2} \sum_i \omega_i(p) |v_i - p|^2 \\
|e(p)| &\leq \frac{c_t}{2} \left({r_{\text{circ}}}^2 - |p - O_{\text{circ}}|^2 \right) \\
|e(p)| &\leq \frac{c_t}{2} {r_{\text{mc}}}^2 \qquad \square
\end{aligned}
$$

Interpolation - Error on the gradient

$$e(p) = f(p) - g(p) \qquad |\nabla e(p)| = |\nabla f(p) - \nabla g(p)|$$
$$e(p) = e_i(p) = -\nabla e(p) \cdot (v_i - p) - \tfrac{1}{2}(v_i - p)^T \mathcal{H}_i \, (v_i - p)$$

$$\begin{aligned}
0 &= e(p) \sum_i \nabla \omega_i = \sum_i e_i(p) \nabla \omega_i \\
&= -\sum_i \left[(v_i - p) \cdot \nabla e(p)\right] \nabla \omega_i - \frac{1}{2} \sum_i \left[(v_i - p)^T \mathcal{H}_i \, (v_i - p)\right] \nabla \omega_i \\
&= \left[p \cdot \nabla e(p)\right] \sum_i \nabla \omega_i - \sum_i \left[v_i \cdot \nabla e(p)\right] \nabla \omega_i - \frac{1}{2} \sum_i \left[(v_i - p)^T \mathcal{H}_i \, (v_i - p)\right] \nabla \omega_i \\
&= -\nabla e(p) - \frac{1}{2} \sum_i \left[(v_i - p)^T \mathcal{H}_i \, (v_i - p)\right] \nabla \omega_i
\end{aligned}$$

Interpolation - Error on the gradient

$$\nabla e(p) = -\frac{1}{2}\sum_i \left[(v_i - p)^T \mathcal{H}_i (v_i - p)\right] \nabla \omega_i$$

$$|\nabla e(p)| \leq \frac{c_t}{2}\sum_i |v_i - p|^2 |\nabla \omega_i| = \frac{c_t}{2}\sum_i \frac{|v_i - p|^2}{a_i}$$

Interpolation - Error on the gradient

$|\nabla e(p)| \leq \frac{c_t}{2} \sum_i \frac{|v_i - p|^2}{a_i}$

Weak bound

$$|\nabla e(p)| \;\leq\; \frac{c_t}{2}\, l_{\mathsf{max}}^2 \sum_i \frac{1}{a_i} = \frac{c_t}{2}\, \frac{l_{\mathsf{max}}^2}{r_{\mathsf{in}}}$$

$V = \frac{1}{d} \sum_i r_{\mathsf{in}} A_i = \frac{1}{d} A_j a_j \implies \frac{1}{a_j} = \frac{1}{r_{\mathsf{in}}} \left(\frac{A_j}{\sum_i A_i} \right)$

$\implies \sum_j \frac{1}{a_j} = \frac{1}{r_{\mathsf{in}}}$

Interpolation - Error on the gradient

$|\nabla e(p)| \leq \frac{c_t}{2} \sum_i \frac{|v_i - p|^2}{a_i}$

the bound is minimum for :

$$
\begin{aligned}
p &= \frac{1}{\sum_i 1/a_i} \sum_j \frac{1}{a_j} v_j \\
&= \frac{1}{\sum_i A_i} \sum_j A_j v_j \qquad\qquad (A_i a_i = dV) \\
&= O_{in} \text{ center of inscribed sphere}
\end{aligned}
$$

Interpolation - Error on the gradient

$O_{in} = \frac{1}{\sum_i A_i} \sum_j A_j v_j$

$$
\begin{aligned}
|\nabla e(O_{in})| &\leq \frac{c_t}{2} \sum_i \frac{|v_i - O_{in}|^2}{a_i} = \frac{c_t}{2dV} \sum_i A_i \left(v_i - \frac{\sum_j A_j v_j}{\sum_m A_m} \right)^2 \\
&= \frac{c_t}{2dV} \sum_i A_i \frac{\left(\sum_j A_j (v_i - v_j) \right)^2}{(\sum_m A_m)^2} \\
&= \frac{c_t}{2dV} \frac{\sum_{i,j,k} A_i A_j A_k (v_i - v_j)(v_i - v_k)}{(\sum_m A_m)^2} \\
&= \frac{c_t}{2dV} \frac{1/2 \sum_{i,j,k} A_i A_j A_k (v_i - v_j)^2}{(\sum_m A_m)^2} \\
&= \frac{c_t}{2dV} \frac{\sum_{i<j} A_i A_j l_{ij}^2}{\sum_m A_m}
\end{aligned}
$$

Interpolation - Error on the gradient

Erreur au point p

$$
\begin{aligned}
|\nabla e(p)| &\leq |\nabla e(O_{in})| + c_t |p - O_{in}| \\
&\leq \frac{c_t}{2dV} \frac{\sum_{i<j} A_i A_j l_{ij}^2}{\sum_m A_m} + c_t \mathrm{max}_i |v_i - O_{in}| \\
&\leq \frac{c_t}{2dV} \frac{\sum_{i<j} A_i A_j l_{ij}^2}{\sum_m A_m} + c_t \mathrm{max}_i \frac{|\sum_{j \neq i} A_j (v_i - v_j)|}{\sum_m A_m} \\
&\leq \frac{c_t}{2dV} \frac{\sum_{i<j} A_i A_j l_{ij}^2}{\sum_m A_m} + c_t \mathrm{max}_i \frac{\sum_{j \neq i} A_j l_{ij}}{\sum_m A_m} \qquad \square
\end{aligned}
$$

Interpolation - Error on the gradient

$$|\nabla e(p)| \leq \frac{c_t}{2dV}\frac{\sum_{i<j} A_i A_j l_{ij}^2}{\sum_m A_m} + c_t \max_i \frac{\sum_{j\neq i} A_j l_{ij}}{\sum_m A_m}$$

A weaker but simpler bound (use $dV \leq A_i l_{ij}$)

$$|\nabla e(p)| \leq \frac{c_t}{2dV}\frac{\sum_{i<j} A_i A_j l_{ij}^2}{\sum_m A_m} + c_t \max_i \frac{\sum_{j\neq i} A_i A_j l_{ij}^2}{dV \sum_m A_m}$$

$$\frac{3c_t}{2dV}\frac{\sum_{i<j} A_i A_j l_{ij}^2}{\sum_m A_m} \qquad \square$$

Interpolation - Error on the gradient

$$|\nabla e(p)| \leq \frac{c_t}{2dV}\frac{\sum_{i<j} A_i A_j l_{ij}^2}{\sum_m A_m} + c_t \max_i \frac{\sum_{j\neq i} A_j l_{ij}}{\sum_m A_m}$$

2D case, $A_i = l_i$ et $l_{ij} = l_k\ k \neq i, j$

$$\begin{aligned}
|\nabla e(p)| &\leq \frac{c_t}{4A}\frac{\sum_{i<j} l_i l_j l_k^2}{\sum_m l_m} + c_t \max_i \frac{\sum_{j\neq i} l_j l_k}{\sum_m l_m} \\
&\leq \frac{c_t}{4A} l_{\text{max}}\, l_{\text{med}}\, l_{\text{min}} + 2c_t \frac{l_{\text{max}} l_{\text{med}}}{\sum_m l_m} \\
&\leq \frac{c_t}{4A} l_{\text{max}}\, l_{\text{med}}\, l_{\text{min}} + \frac{2c_t}{2A} l_{\text{max}} l_{\text{med}} r_{\text{in}} \\
&\leq \frac{c_t}{4A} l_{\text{max}}\, l_{\text{med}}\, (l_{\text{min}} + 4r_{\text{in}}) \qquad \square
\end{aligned}$$

Finite element

Example 1 : Poisson equation

$$
\begin{aligned}
-\nabla^2 f(p) = &\ \eta(p) \qquad \forall p \in \text{domain } \Omega \\
f(p) = &\ 0 \qquad \forall p \in \Gamma \text{ boundary of } \Omega
\end{aligned}
$$

Weak formulation : for any function v that vanishes on Γ

$$
\begin{aligned}
\iint_\Omega \left[-\nabla^2 f(p) - \eta(p)\right] v(p) d^2 p &= 0 \\
\iint_\Omega \left[\nabla f(p) \cdot \nabla v(p) - \eta(p) v(p)\right] d^2 p &= 0
\end{aligned}
$$

integration per parts $\qquad \left(\nabla^2 f\right) v = \nabla \cdot (v \nabla f) - \nabla f \cdot \nabla v$

Divergence theorem $\qquad \iint_\Omega \nabla \cdot \mathbf{u} = \int_\Gamma \mathbf{u} \cdot \mathbf{n}$

Finite elements

Galerkin method

1. choose a finite space of function $E_n = \{u_1, u_2, \ldots, u_n\}$
2. approximation of $f(p)$ in E_n, $h(p) = \sum_j h_j u_j(p)$
3. weak formulation using test functions $v(p) \in E_n$

$$
\begin{aligned}
\iint_\Omega \left[\nabla f(p) \cdot \nabla v(p) - \eta(p) v(p)\right] d^2 p &= 0 \\
K_{ij} h_j &= \eta_i \\
K_{ij} &= \iint_\Omega \nabla u_i(p) \cdot \nabla u_j(p) d^2 p \\
\eta_i &= \iint_\Omega \eta(p) u_i(p) d^2(p)
\end{aligned}
$$

Finite elements

Example 2 :

$$
\begin{aligned}
-\nabla^2 f(p) &= \eta(p) \quad \forall p \in \Omega \;\; \Omega \\
\nabla f(p) \cdot n(p) + \beta(p) f(p) &= \gamma(p) \quad \forall p \in \Gamma \;\; \partial\Omega
\end{aligned}
$$

Weak formulation

$$
\begin{aligned}
\iint_\Omega -\nabla^2 f(p)\, v(p)\, d^2p &= \iint_\Omega \eta(p) v(p)\, d^2p \\
\iint_\Omega \nabla f \cdot \nabla v \, d^2p - \int_\Gamma v\, \nabla f \cdot n \, dp &= \iint_\Omega \eta v \, d^2p \\
\iint_\Omega \nabla f \cdot \nabla v \, d^2p + \int_\Gamma \beta f v \, dp &= \iint_\Omega \eta v \, d^2p + \int_\Gamma \gamma v \, dp
\end{aligned}
$$

Finite elements - Example2

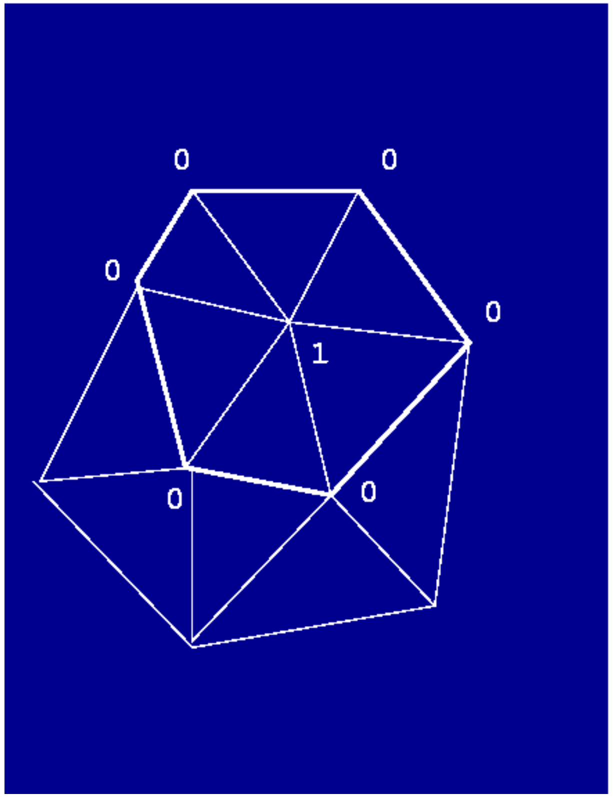
Galerkin method

1. choose a finite space of function $E_n = \{u_1, u_2, \ldots, u_n\}$
2. approximation of $f(p)$ in E_n, $h(p) = \sum_j h_j u_j(p)$
3. weak formulation using test functions $v(p) \in E_n$

$$
\begin{aligned}
\iint_\Omega \nabla f \cdot \nabla v \, d^2p + \int_\Gamma \beta f v \, dp &= \iint_\Omega \eta v \, d^2p + \int_\Gamma \gamma v \, dp \\
K_{ij} h_j &= \eta_i \\
K_{ij} &= \iint_\Omega \nabla u_i \cdot \nabla u_j \, d^2p + \int_\Gamma \beta u_i u_j \, dp \\
\eta_i &= \iint_\Omega \eta u_i \, d^2p + \int_\Gamma \gamma u_i \, dp
\end{aligned}
$$

Choosing $E_n = \{u_1, u_2, \ldots, u_n\}$

- h(p) has to accurately approximate f(p)
- K_{ij} and η_i should be easy to compute
- K must be a sparse, well conditioned matrix



Finite elements of type P1

Mesh $T(\Omega)$,

u_i piecewise linear, $u_i(p_j) = \delta_{ij}$

$u_j(p) = w_j(p)$ if $p \in t \in \text{star}(p_j)$

$= 0$ otherwise

$h(p) = \sum h_i u_i(p)$ is piecewise linear.

For $p \in t(p_1 p_2 p_3)$, $h(p) = h_1 w_1(p) + h_2 w_2(p) + h_3 w_3(p)$

Finite elements

Othe types of finite elements

- Linear element in dimension 1

  $u(x) = 1 - |x| \qquad h(x) = a + bx$

- Cubic element in dimension 1

  $u(x) = (x^2 - 1)(2x + 1), (x - 1)^2 x$

  $h(x) = a + bx + cx^2 + dx^3$

- Type Q1 : Bilinear on a rectangle

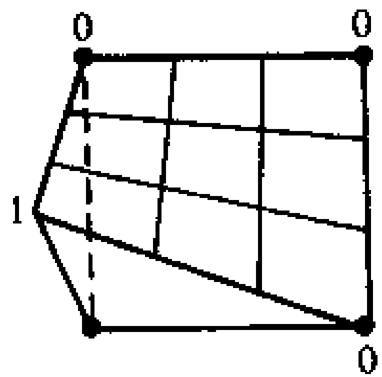
  $u(x, y) = 1 - x - y + xy$

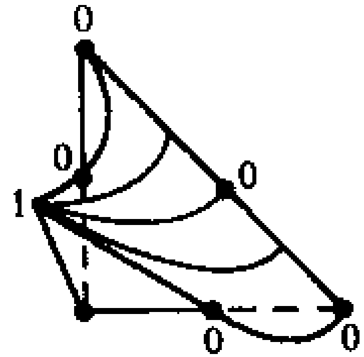
  $h(x, y) = a + bx + cy + dxy$

- Type P2 : Quadratics on a triangle

  $u(x, y) = (1 - x - y)(1 - 2x - 2y)$

  $h(x, y) = a + bx + cy + dx^2 + exy + fy^2$





Finite element - Error analysis

1. Solving the linear system
   - iterative methods (Jacobi, conjugate gradient)
   - direct methods (Gauss elimination)

   In any case, the error depends on conditioning κ
   of the global stiffness matrix K_{ij}

$$\kappa = \frac{\lambda^K_{\text{max}}}{\lambda^K_{\text{min}}} \qquad \lambda^K_{\text{max}}, \lambda^K_{\text{min}}$$

   min and max of K eigenvalues

2. Discretization error
   related to the search of a solution in the finite function space E_n

Finite elements - Stiffness matrix K_{ij}

$$\text{Poisson eq.} \qquad K_{ij} = \iint_\Omega \nabla u_i(p) \nabla u_j(p) d^2 p$$

$K_{ij} = 0$ except if p_i and $p_j \in$ the same cell of the mesh.
Contribution of each mesh triangle du maillage to K_{ij}.
$t = p_1 p_2 p_3$ contributes to $K_{11}, K_{22}, K_{33}, K_{12}, K_{13}, K_{23}$.
For linear elements P1, the contribution K_t of $t = p_1 p_2 p_3$ is

$$K_t = A \begin{bmatrix} \nabla\omega_1 \cdot \nabla\omega_1 & \nabla\omega_1 \cdot \nabla\omega_2 & \nabla\omega_1 \cdot \nabla\omega_3 \\ \nabla\omega_2 \cdot \nabla\omega_1 & \nabla\omega_2 \cdot \nabla\omega_2 & \nabla\omega_2 \cdot \nabla\omega_3 \\ \nabla\omega_3 \cdot \nabla\omega_1 & \nabla\omega_3 \cdot \nabla\omega_2 & \nabla\omega_3 \cdot \nabla\omega_3 \end{bmatrix}$$

Finite elements - The stiffness matrix K_{ij}

$$K_t = \frac{1}{8A}\begin{bmatrix} 2l_1^2 & l_3^2 - l_1^2 - l_2^2 & l_2^2 - l_1^2 - l_3^2 \\ l_3^2 - l_1^2 - l_2^2 & 2l_2^2 & l_1^2 - l_1^2 - l_3^2 \\ l_2^2 - l_1^2 - l_3^2 & l_1^2 - l_1^2 - l_3^2 & 2l_3^2 \end{bmatrix}$$

$$K_t = \frac{1}{2}\begin{bmatrix} \cot\theta_2 + \cot\theta_3 & -\cot\theta_3 & -\cot\theta_2 \\ -\cot\theta_3 & \cot\theta_3 + \cot\theta_1 & -\cot\theta_1 \\ -\cot\theta_2 & -\cot\theta_1 & \cot\theta_1 + \cot\theta_2 \end{bmatrix}$$

Finite elements - Conditioning of the stiffness matrix K_{ij}

$$\kappa = \frac{\lambda^K_{\text{max}}}{\lambda^K_{\text{min}}}$$

λ^K_{min} depends on the equation and on elements size
lower bound proportional to the surface (volume)
of the smallest element

λ^K_{max} can be made arbitrarily large
by a single bad element

m max number of cells incident to a vertex

λ^t_{max} max eigenvalue of K_t

$$\text{max}_t\, \lambda^t_{\text{max}} \leq \lambda^K_{\text{max}} \leq m \text{max}_t\, \lambda^t_{\text{max}}$$

Finite elements - Conditioning of the stiffness matrix K_{ij}

Poisson equation

$$\lambda^t = \frac{l_1^2 + l_2^2 + l_3^2 \pm \sqrt{(l_1^2 + l_2^2 + l_3^2)^2 - 48A^2}}{8A}$$

$$\frac{l_1^2 + l_2^2 + l_3^2}{8A} \leq \lambda^t_{\text{max}} \leq \frac{l_1^2 + l_2^2 + l_3^2}{4A}$$

bad triangle : small area $\Longleftrightarrow$ large λ^t_{max}
small angles ruin the condition number of the stiffness matrix
the upper bound for λ^t_{max} is scale invariant
If there is no small angles,
the lower bound for $\lambda^K_{\text{min}}$is $\propto A_{\text{min}}$
uniform sizing mesh $\kappa \propto O(1/l^2) = n$ nb of mesh elements

Finite elements - Discretization error

Discretization error : related to interpolation error

but depends on PDE

f exact solution of PDE

h solution obtained by finite elements

g linear interpolation of h on the mesh, $h \neq g$

For some PDE, the finite elements solution minimizes an *energy function*. For Poisson equation, h minimizes

$$\|f - h\|_{H^1(\Omega)} = \left(\iint_\Omega \left((f-h)^2 + |\nabla f - \nabla h|^2) \right) d^2p \right)^{1/2}$$

Finite elements - Discretization error

Because h is optimal for this energy,

$$\begin{aligned} \|f-h\|_{H^1(\Omega)} &\leq \|f-g\|_{H^1(\Omega)} \\ &\leq \left(\sum_{t\in T} V_t \left(\|f-g\|_{\infty(t)} + \|\nabla f - \nabla g\|^2_{\infty(t)}\right)\right)^{1/2} \end{aligned}$$

Anisotropy

Interpolation : anisotropic curvature
Finite elements : anisotropic PDE
The optimal mesh is anisotropic

Anisotropic curvature tensor
$H(p)$ Hessian of $f(p)$, $\forall\, \boldsymbol{d} \quad |\boldsymbol{d}^T H(p) \boldsymbol{d}| \leq \boldsymbol{d}^T C_t \boldsymbol{d}$

$$C_t = \xi_1 \boldsymbol{v_1} \boldsymbol{v_1}^T + \xi_2 \boldsymbol{v_2} \boldsymbol{v_2}^T + \xi_3 \boldsymbol{v_3} \boldsymbol{v_3}^T$$

Transformation $\widehat{p} = Ep$

$$\begin{aligned} E &= \sqrt{\xi_1/\xi_{\text{max}}}\,\boldsymbol{v_1}\boldsymbol{v_1}^T + \sqrt{\xi_2/\xi_{\text{max}}}\,\boldsymbol{v_2}\boldsymbol{v_2}^T + \sqrt{\xi_3/\xi_{\text{max}}}\,\boldsymbol{v_3}\boldsymbol{v_3}^T \\ E^2 &= \frac{1}{\xi_{\text{max}}} C_t \end{aligned}$$

Anisotropie

$\widehat{f}(q) = f(E^{-1}q) \qquad \widehat{f}(\widehat{q}) = f(q)$
$\widehat{g}(q) = g(E^{-1}q)$

$\widehat{f}$ has an istopic curvature bound

$$\begin{aligned}
\widehat{f}''_{\mathbf{d}} &= \frac{d^2}{d\alpha_2} f(E^{-1}(q + \alpha\mathbf{d}))\Big|_{\alpha=0} \\
&= (E^{-1}\mathbf{d})^T H(q)(E^{-1}\mathbf{d}) \\
&\leq \mathbf{d}^T E^{-1} C_t E^{-1}\mathbf{d} = c_t|\mathbf{d}|^2
\end{aligned}$$

Bound on interpolation error $\|f - g\|_{\infty(t)} = \|\widehat{f} - \widehat{g}\|_{\infty(\widehat{t})}$